Quantum Field Theory II

Winter Term 2020/21

by M. Zirnbauer

- Perturbation Theory
  - Asymptotic series and Borel resummation
  - Feynman graphs for phi^4 theory
  - Legendre transform to vertex functions
  - QM and QFT on multiply connected spaces
  - One-loop processes in QED
- Symmetry breaking and collective phenomena
  - Hartree-Fock-Bogoliubov mean-field ground states
  - BCS theory of superconductors and superfluids
  - Goldstone theorem
  - Ginzburg-Landau theory
  - Anderson-Higgs mechanism
  - Tutorial: gauge invariance
  - H-picture of superconductivity
- Renormalization
  - Decimation, Kadanoff block spin transformation
  - Migdal-Kadanoff approximation
  - Universality and scaling
  - Kosterlitz-Thouless phase transition: RG treatment
  - Vertex fcts & effective actions: 1-loop approximation
  - LGW mean field theory: Ginzburg criterion
  - Background field RG of nonlinear sigma models
  - Functional determinants by heat-kernel method

- Old stuff (not taught in WS 2020/21):
  Gauge theories of quantum matter
  - Chains and co-chains
  - Ising model and Ising gauge theory
  - Duality transformations (Kramers-Wannier, etc.)
  - Wegner-Wilson loop
  - U(1) lattice gauge theory in 3D
  - Boson-vortex duality
  - Hamiltonian formulation
  - Toric code
  - Nonabelian gauge theory

Chapter I : Perturbation Theory

I.1 Preparation: a simple example

Toy model partition function:

$$Z(t) := \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} dx \; e^{-\frac{x^2}{4t} - x^4} \qquad (t > 0).$$

Normalization: $\lim_{t \to 0+} Z(t) = 1$.

Equivalent expression: $Z(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dy \; e^{-y^2 - (4t)^2 y^4} \qquad (x = \sqrt{4t}\, y)$.

Assuming that t is small, perturbation theory attempts to compute $Z(t)$ by an expansion in powers of t. Is this going to work?

Computational trick (in case you've lost your field theory notes on Gaussian integrals):

$$\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} = e^{t\Delta} \delta(x), \qquad \Delta = \frac{d^2}{dx^2} \qquad \text{(proof by Fourier transform)}.$$

Hence $Z(t) = \int_{-\infty}^{\infty} dx \; e^{-x^4} e^{t\Delta} \delta(x)$. By partial integration it follows that

$$Z(t) = \int_{-\infty}^{\infty} dx \; \delta(x)\, e^{t\Delta} e^{-x^4} = \left(e^{t\Delta} e^{-x^4} \right)\Big|_{x=0} .$$

Interpretation:
1) initial distribution function $f(x,0) = e^{-x^4}$,
2) run diffusive dynamics $\frac{\partial}{\partial t} f(x,t) = \frac{\partial^2}{\partial x^2} f(x,t)$,
3) observe $f(0,t) = Z(t)$.

Does $Z(t)$ have an expansion in powers of t?

$$Z(t) \overset{?}{=} \sum_{m=0}^{\infty} \frac{t^m}{m!} \Delta^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{4n} \Big|_{x=0} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{(2n)!\,n!} \Delta^{2n} x^{4n}$$

$$= \sum_{n=0}^{\infty} \frac{(4n)!}{(2n)!\,n!} (-t^2)^n .$$

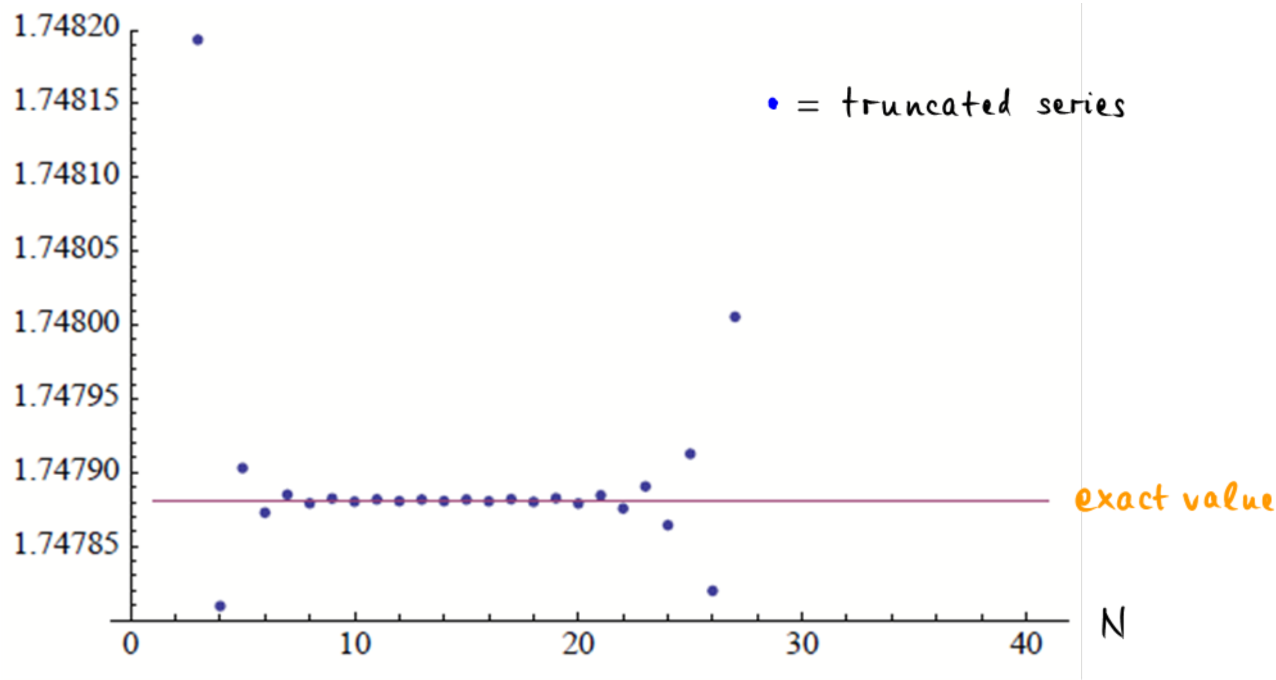
This series is divergent [use Stirling's formula $n! \approx \sqrt{2\pi n}\, (n/e)^n$].

Thus $Z(t)$ is not an analytic function of t at the point $t = 0$.

Nevertheless, the series is not useless! If one truncates it at order N and takes t to be small enough, then the series is numerically good for a range of N values.

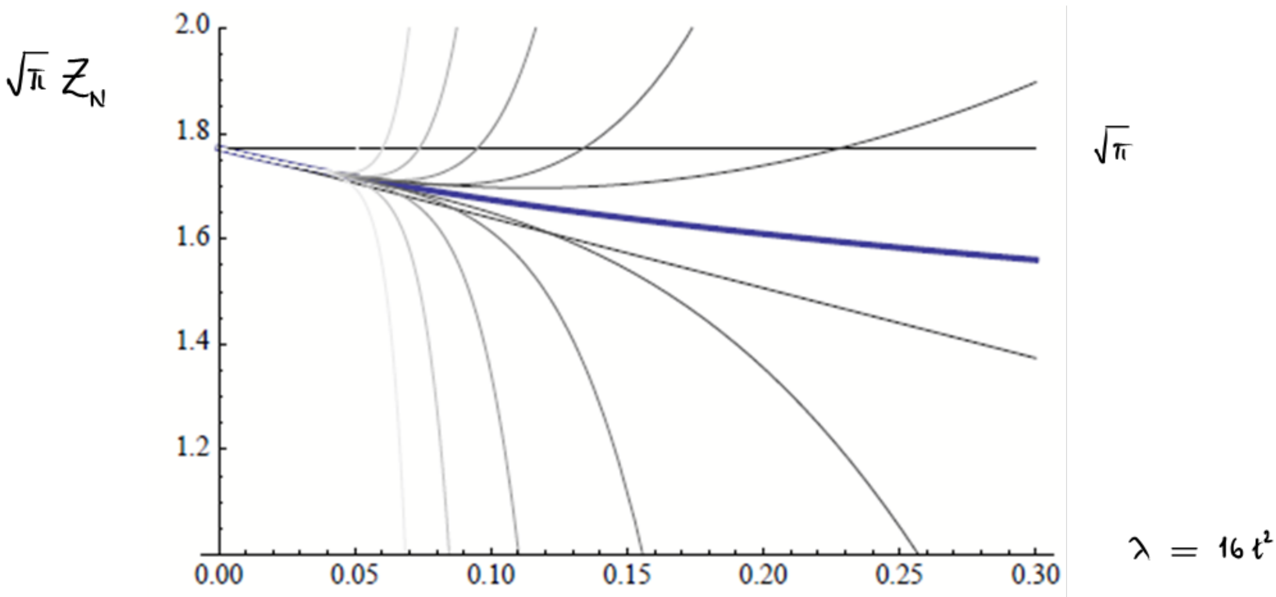
For example:

$\sqrt{\pi}\ Z(t)$ for $t = 0.035$



(taken from arXiv: 1201.2714)

Here is how the truncated series (for different values of N) compares with the true function $Z(t)$:



I.2 Asymptotic series and Borel resummation

Begin with background/motivation: Laplace-Borel transform and its inverse.

Let $Z(p) = \sum_{k=0}^{N} a_k p^k$ be a polynomial in $p \in \mathbb{C}$.

Laplace transform: $\hat{Z}(q) = \int_0^\infty Z(p/q)\, e^{-p} dp = \sum_{k=0}^{N} k!\, a_k q^{-k}$ $(q \neq 0)$.

Inverse transform: $Z(p) = (2\pi i)^{-1} \oint_{U_1} \hat{Z}(q)\, e^{pq} q^{-1} dq = \sum_{k=0}^{N} a_k p^k$.

Notice: the coefficients entering the two sums differ by a factorial.

And our divergent series for $Z(t)$ will acquire a finite radius of convergence if we divide the coefficient of t^{2n} by $(2n)!$. Indeed,

$$\frac{1}{(2n)!}\cdot\frac{(4n)!}{(2n)!\,n!} = \binom{4n}{2n}\cdot\frac{1}{n!} \overset{n \text{ large}}{\approx} \frac{2^{4n}}{\sqrt{2\pi n}}\cdot\frac{1}{n!}$$

Some mathematical background.

Definition: a series $\sum_{n=0}^{\infty} a_n z^n$ is called asymptotic to a function f (assumed to exist on $\mathbb{R}_+$) as $z \to 0+$ if

$$\forall N \in \mathbb{N}: \quad \lim_{z\to 0+} \frac{f(z) - \sum_{n=0}^{N} a_n z^n}{z^N} = 0.$$

Remark. f can have at most one asymptotic series.

Question: What about the converse? (Is there at most one function per asymptotic series?)

Definition: Let f be analytic on $S_\varepsilon = \left\{ z \in \mathbb{C} \,\middle|\, |z| < R, \ |\arg z| < \frac{\pi}{2} + \varepsilon \right\}$ $(\varepsilon > 0)$.

Then $\sum_{n=0}^{\infty} a_n z^n$ is called a strongly asymptotic series if

$$\exists\, C, \sigma: \quad \forall N \in \mathbb{N},\ z \in \overline{S}_\varepsilon: \quad \left| f(z) - \sum_n^N a_n z^n \right| \le C\sigma^{N+1}(N+1)!\,|z|^{N+1}.$$

Fact. If $\sum_{n=0}^{\infty} a_n z^n$ is strongly asymptotic to f and g, then $f = g$.

Our example. The asymptotic series $\sum_{n=0}^{\infty} \frac{(4n)!}{(2n)!\,n!}(-t^2)^n$ is strongly asymptotic to the function

$$\tilde{Z}(t) = \int_0^\infty dt'\, e^{-t'} \sum_{n=0}^{\infty} \frac{(4n)!}{(2n)!\,n!}\,\frac{(-tt')^{2n}}{(2n)!}.$$

Hence

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{4t} - x^4} = Z(t) = \tilde{Z}(t) = \int_0^\infty dt'\, e^{-t'} \sum_{n=0}^{\infty} \frac{(4n)!}{(2n)!\,n!}\,\frac{(-tt')^{2n}}{(2n)!}.$$

Q: Can one anticipate that the perturbation series for $Z(t)$ turns out to be divergent? A: Yes!

Dyson-argument: substitute $x^2 \to x^2\cdot 4t$ and let $(4t)^2 \equiv \lambda$. Then

$Z(\sqrt{\lambda}/4) \equiv F(\lambda) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2 - \lambda x^4}$. $\lambda = 0$ cannot be a point of analyticity of $F(\lambda)$, as the integral fails to exist for any negative λ (no matter how close to zero).

I.3 ϕ^4 theory

Scalar field $\phi : \mathbb{R}^d \supset \Lambda \longrightarrow \mathbb{R}$.

Volume form d^dx. Metric on $\mathbb{R}^d$ $\rightsquigarrow$ $\operatorname{grad}\phi \equiv \nabla\phi$ (gradient).

Action functional (in Euclidean signature):

$$S[\phi] = \int_\Lambda d^dx \left(\tfrac{1}{2} |\operatorname{grad}\phi|^2 + V(\phi) \right), \qquad V(\phi) = \tfrac{r}{2}\phi^2 + g\,\phi^4 .$$

Functional integral (partition function) $\quad Z = \int \mathcal{D}\phi \; e^{-S[\phi]}$.

Motivation. Ising model, spins $\sigma = \pm 1$ $\longleftrightarrow$

For negative values of r the function $V(\phi)$ has the form of a symmetric double well.

The two minima of $V(\phi)$ are energetically preferred values of the field ϕ.

Thus it seems plausible that ϕ^4 theory (in the parameter range of $r < 0$) arises as a continuum approximation to the Ising model. (Ferromagnetic coupling of the Ising spins corresponds to the gradient term of the ϕ^4 theory.

For quantitative details, see Altland & Simons.

In the following, however, ϕ^4 theory will be considered in the parameter range of $r > 0$.

To set up the **perturbation expansion**, we follow the scheme introduced for the simple example of Section I.1. Recall (with slightly adjusted conventions)

$$\frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}} = \int \frac{dk}{2\pi}\, e^{-\frac{t}{2}k^2 + ikx} = e^{\frac{t}{2}\frac{\partial^2}{\partial x^2}} \int \frac{dk}{2\pi}\, e^{ikx} = e^{\frac{t}{2}\frac{\partial^2}{\partial x^2}}\, \delta(x) .$$

Here is the same calculation in the field-theoretic context:

by partial integration, assuming $\phi\big|_{\partial\Lambda} = 0$.

$$e^{-\frac{1}{2}\int_\Lambda d^dx\, (|\nabla\phi|^2 + m^2\phi^2)} = e^{-\frac{1}{2}\int_\Lambda d^dx\, \phi\,(-\nabla^2 + m^2)\,\phi}$$

$$= \text{const} \int \mathcal{D}\xi \; e^{-\frac{1}{2}\int_\Lambda d^dx\, \xi\,(-\nabla^2+m^2)^{-1}\,\xi \; + \; i\int_\Lambda d^dx\, \xi\phi}$$

do the Gaussian integral by completing the square

functional derivative $\quad \frac{\delta}{\delta\phi(x)} \int d^dx\, \xi\phi = \xi(x)$

$$= \text{const}\; e^{\frac{1}{2}\int_\Lambda d^dx\, \frac{\delta}{\delta\phi}(-\nabla^2+m^2)^{-1}\frac{\delta}{\delta\phi}} \int \mathcal{D}\xi \; e^{i\int_\Lambda d^dx\, \xi\phi}$$

$$= \text{const}' \, e^{\frac{1}{2}\int_\Lambda d^dx \, \frac{\delta}{\delta\phi}(-\nabla^2+m^2)^{-1}\frac{\delta}{\delta\phi}} \, \delta[\phi] .$$

Dirac distribution, supported on the zero field $\phi(x) \equiv 0$.

The resulting expression can be written in the form

$$e^{-\frac{1}{2}\int_\Lambda d^dx \, (|\nabla\phi|^2 + m^2\phi^2)} = Z_{\text{free}} \cdot e^{\frac{1}{2}\int_\Lambda d^dx \int_\Lambda d^dy \, \frac{\delta}{\delta\phi(x)} G_0(x,y) \frac{\delta}{\delta\phi(y)}} \, \delta[\phi] , \quad (*)$$

$$\text{where } G_0(x,y) = (-\nabla^2+m^2)^{-1}(x,y) \underset{\Lambda = \mathbb{R}^d}{=} \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik(x-y)}}{k^2+m^2} \sim |x-y|^{-d+2} e^{-m|x-y|}$$

is the propagator of the free theory (called the "free propagator" for short).

Z_{free} is the partition function of the Gaussian (or free) theory:

$$Z_{\text{free}} = \int \mathcal{D}\phi \; e^{-\frac{1}{2}\int_\Lambda d^dx \, (|\nabla\phi|^2 + m^2\phi^2)} .$$

To apply the formula $(*)$ to the full (interacting) theory, let

$$V(\phi) = \tfrac{1}{2} m^2 \phi^2 + \tilde{V}(\phi) , \qquad r = m^2 .$$

(In the present case of ϕ^4 theory we have $\tilde{V}(\phi) = g\phi^4$.) Then

$$Z/Z_{\text{free}} = \int \mathcal{D}\phi \; e^{-\int_\Lambda d^dx \, \tilde{V}(\phi)} \; e^{\frac{1}{2}\int_\Lambda d^dx \int_\Lambda d^dy \, \frac{\delta}{\delta\phi(x)} G_0(x,y) \frac{\delta}{\delta\phi(y)}} \, \delta[\phi] ,$$

$$= e^{\frac{1}{2}\int_\Lambda d^dx \int_\Lambda d^dy \, \frac{\delta}{\delta\phi(x)} G_0(x,y) \frac{\delta}{\delta\phi(y)}} \; e^{-\int_\Lambda d^dx \, \tilde{V}(\phi)} \Big|_{\phi \equiv 0} .$$

Expansion to first order in the coupling g:

$$Z/Z_{\text{free}} = 1 + \frac{1}{2!}\left(\frac{1}{2}\right)^2 \iiiint_{\Lambda^4} d^dx_1 d^dx_2 d^dx_3 d^dx_4 \, G_0(x_1,x_2) G_0(x_3,x_4) \frac{\delta}{\delta\phi(x_1)} \frac{\delta}{\delta\phi(x_2)} \frac{\delta}{\delta\phi(x_3)} \frac{\delta}{\delta\phi(x_4)} \left(-g \int_\Lambda d^dx \, \phi(x)^4 \right) + \ldots$$

$$= 1 + \frac{1}{2!}\left(\frac{1}{2}\right)^2 4! \, (-g) \int_\Lambda d^dx \, G_0(x,x)^2 + \ldots$$

$$\text{Now } G_0(x,x) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik(x-x)}}{k^2+m^2} = \frac{1}{(2\pi)^d} \int \frac{d^dk}{k^2+m^2} = \;? \qquad (d \geq 2)$$

This divergence (for $d \geq 2$) is called an ultraviolet (UV) divergence because it is due to large values of k.

To remove UV divergences, one introduces a regularization scheme.

One of several options is UV regularization by discretization on a lattice.

Recall a fact from the theory of the Fourier transform / Fourier series :

If position space is $\mathbb{Z}\cdot a$ (lattice with lattice constant a), then momentum space is $\mathbb{R}/2\pi a\mathbb{Z}$, e.g. realized by the interval $\left[-\frac{\pi}{a}, +\frac{\pi}{a}\right]$.

Discretization of second derivative: $-f''(x) \approx \big(-f(x+a) + 2f(x) - f(x+a)\big)/a^2$

In Fourier/momentum space the (negative) second derivative becomes multiplication by

$$\frac{1}{a^2}\left(2 - e^{ika} - e^{-ika}\right) = \frac{2}{a^2}\left(1-\cos ka\right) \approx k^2 \quad (\text{for } ka \text{ small}).$$

Generalization to d dimensions: $\varepsilon(k) := \frac{2}{a^2}\sum_{j=1}^{d}\left(1-\cos(k_j a)\right) \approx |k|^2$.

Lattice-regularized free propagator: $G_0(x,y) = \displaystyle\int_{\left[-\frac{\pi}{a},+\frac{\pi}{a}\right]^d} \frac{d^dk}{(2\pi)^d}\,\frac{e^{ik(x-y)}}{\varepsilon(k)+m^2}$.

On the diagonal: $G_0(x,x) = \displaystyle\int_{\left[-\frac{\pi}{a},+\frac{\pi}{a}\right]^d} \frac{d^dk}{(2\pi)^d}\,\frac{1}{\varepsilon(k)+m^2} < \infty$.

Other schemes: Pauli-Villars regularization

dimensional regularization

heat kernel / zeta function regularization.

Up to now we have been addressing the partition function. More informative are the n-point functions: $\langle \phi(x_1)\,\phi(x_2)\,\cdots\,\phi(x_n)\rangle$ where

$$\langle A\rangle := Z^{-1}\int \mathcal{D}\phi\; A\; e^{-S[\phi]}.$$

To express the n-point function (in perturbation theory) as a multiple derivative, we introduce the generating functional

$$Z[j] := \int \mathcal{D}\phi\; e^{-S[\phi]+\int d^dx\, j(x)\,\phi(x)}.$$

Making the same steps as before, we obtain

$$Z[j] = Z_{free}\cdot\int \mathcal{D}\phi\; e^{\int d^dx\,(j\phi-\tilde{V}(\phi))}\; e^{\frac{1}{2}\int d^dx\int d^dy\,\frac{\delta}{\delta\phi(x)}G_0(x,y)\frac{\delta}{\delta\phi(y)}}\;\delta[\phi],$$

$$= Z_{free}\cdot e^{\frac{1}{2}\int d^dx\int d^dy\,\frac{\delta}{\delta\phi(x)}G_0(x,y)\frac{\delta}{\delta\phi(y)}}\; e^{\int d^dx\,(j\phi-\tilde{V}(\phi))}\Big|_{\phi\equiv 0}.$$

The n-point functions are now generated by taking derivatives with respect to the sources j at $j=0$. In particular, for the two-point function one gets

$$\langle\phi(x_1)\,\phi(x_2)\rangle = Z^{-1}\,\frac{\delta}{\delta j(x_1)}\,\frac{\delta}{\delta j(x_2)}\,Z[j]\Big|_{j=0}$$

$$= (Z_{free}/Z)\cdot e^{\frac{1}{2}\int d^dx\int d^dy\,\frac{\delta}{\delta\phi(x)}G_0(x,y)\frac{\delta}{\delta\phi(y)}}\;\phi(x_1)\,\phi(x_2)\;e^{-\int d^dx\,\tilde{V}(\phi)}\Big|_{\phi\equiv 0}.$$

Lowest order $(g=0)$: $\quad Z_{free}/Z = 1$,

$$\langle\phi(x_1)\,\phi(x_2)\rangle = e^{\frac{1}{2}\int d^dx\int d^dy\,\frac{\delta}{\delta\phi(x)}G_0(x,y)\frac{\delta}{\delta\phi(y)}}\;\phi(x_1)\,\phi(x_2)\Big|_{\phi\equiv 0} = G_0(x_1,x_2).$$

First order (in g). Let $D \equiv \frac{1}{2}\int d^dx\int d^dy\,\frac{\delta}{\delta\phi(x)}G_0(x,y)\frac{\delta}{\delta\phi(y)}$ and $S_{int}\equiv\int d^dx\,\tilde{V}(\phi)$.

Then $Z_{free}/Z = \left(e^{D}\cdot e^{-S_{int}}\right)^{-1}\Big|_{\phi=0} = \left(1-\frac{1}{2}D^2S_{int}+\ldots\right)^{-1}\Big|_{\phi=0} = 1+\frac{1}{2}D^2S_{int}+\ldots$

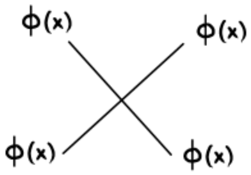
$$e^{D}\left(\phi(x_1)\phi(x_2)\,e^{-S_{int}}\right)\Big|_{\phi=0} = D\left(\phi(x_1)\phi(x_2)\right) + \frac{D^3}{3!}\left(\phi(x_1)\phi(x_2)(-S_{int})\right)+\ldots$$

$$= \left(D\cdot\phi(x_1)\phi(x_2)\right)\left(1-\frac{D^2}{2!}S_{int}+\ldots\right) + \text{"non-vacuum graphs"}$$

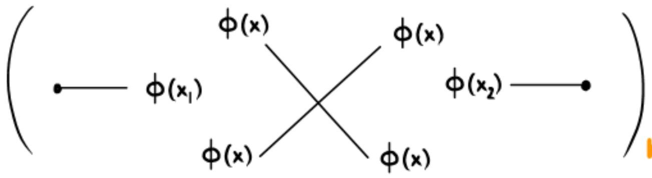
Hence $\quad \langle\phi(x_1)\,\phi(x_2)\rangle = D\left(\phi(x_1)\phi(x_2)\right) - \frac{D^3}{3!}\left(\phi(x_1)\phi(x_2)\,S_{int}\right)_{n.v.}+\ldots$

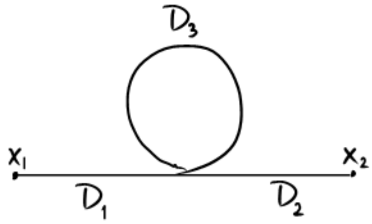
Lecture 03

Graphical notation.

$g \int d^dx\, \phi(x)^4 \rightsquigarrow g$ 

"interaction vertex"

$\frac{D^3}{3!}$ 

n.v. $= \frac{3!}{3!}\, 4 \cdot 3 \cdot$ 

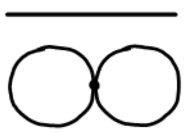
$$= 12 \int d^dx\; D\big(\phi(x_1)\phi(x)\big)\; D\big(\phi(x_2)\phi(x)\big)\; D\big(\phi(x)\phi(x)\big)\,.$$

Hence $\langle \phi(x_1)\,\phi(x_2)\rangle = G_0(x_1,x_2) - 12\, g \int d^dx\; G_0(x_1,x)\, G_0(x_2,x)\, G_0(x,x) + \mathcal{O}(g^2)$.

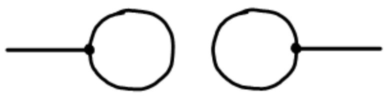
Vacuum vs. non-vacuum graphs.

A vacuum graph (in the context of perturbation theory for the n-point function) is a graph not all parts of which are connected to an external line.

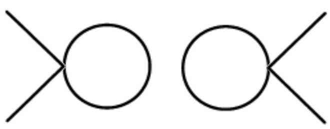
Examples:



vacuum



non-vacuum (ϕ^3 theory)



4-point function

Fact. n-point functions are sums of non-vacuum graphs (in standard language: of graphs not containing vacuum subgraphs).

<u>Proof.</u> Let $X[\phi] \equiv \phi(x_1)\,\phi(x_2)\cdots\phi(x_n)$, D as before.

$$\langle X[\phi]\rangle = \frac{Z_{free}}{Z}\, e^{D}\left(X[\phi]\, e^{-S_{int}}\right)\Big|_{\phi\equiv 0}.$$

$$e^{D}\left(X\, e^{-S_{int}}\right)\Big|_{\phi=0} = \sum_{n=0}^{\infty}\frac{D^n}{n!}\left(X\, e^{-S_{int}}\right)\Big|_{\phi=0}$$

vacuum (sub)graphs (not connected to X)

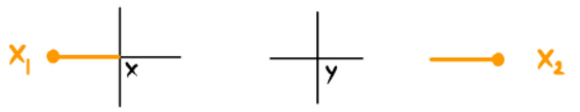
$$= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{p=0}^{n}\binom{n}{p} D^{n-p}\left(X\, e^{-S_{int}}\right)\Big|_{\phi=0}^{n.v.}\cdot D^{p}\left(e^{-S_{int}}\right)\Big|_{\phi=0}$$

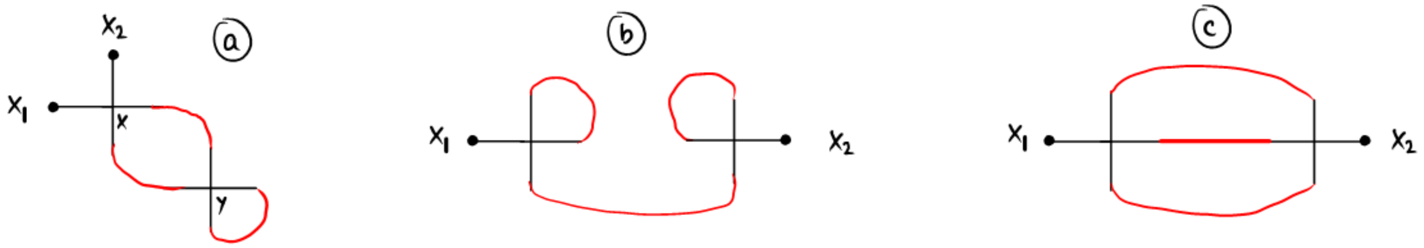
$$= \sum_{m=0}^{\infty}\frac{D^m}{m!}\left(X\, e^{-S_{int}}\right)\Big|_{\phi=0}^{n.v.}\cdot \underbrace{\sum_{p=0}^{n}\frac{D^p}{p!}\left(e^{-S_{int}}\right)\Big|_{\phi=0}}_{=\,Z/Z_{free}}.$$

new summation variable $m = n-p$

Hence $$\langle X[\phi]\rangle = e^{D}\left(X[\phi]\, e^{-S_{int}}\right)\Big|_{\phi=0}^{n.v.}$$ ∎

Contribution of **second order** (in g) to $\langle \phi(x_1)\,\phi(x_2)\rangle$:

starting point: 



Taking into account the combinatorial factors one obtains

$$ⓐ = 4\cdot 3\cdot\binom{4}{2}\cdot 2\cdot g^2\int d^dx\int d^dy\; G_0(x_1,x)\,G_0(x_2,x)\,G_0(x,y)^2\,G_0(y,y),$$

$$ⓑ = 4^2\cdot 3^2\cdot g^2\int d^dx\int d^dy\; G_0(x_1,x)\,G_0(x,x)\,G_0(x,y)\,G_0(y,y)\,G_0(y,x_2),$$

$$ⓒ = 4^2\cdot 3!\cdot g^2\int d^dx\int d^dy\; G_0(x_1,x)\,G_0(x,y)^3\,G_0(y,x_2).$$

For a translation-invariant system it is advantageous to Fourier-transform to the momentum representation:

$$G_0(x,y) = \int \frac{d^dk}{(2\pi)^d}\, \tilde{G}_0(k)\, e^{ik(x-y)}.$$

$$ⓐ = 144\, g^2 \int d^dx \int d^dy \int \frac{d^dk_1}{(2\pi)^d}\, \tilde{G}_0(k_1)\, e^{ik_1(x_1-x)} \int \frac{d^dk_2}{(2\pi)^d}\, \tilde{G}_0(k_2)\, e^{ik_2(x_2-x)}$$

$$\cdot\, G_0(0,0) \int \frac{d^dk}{(2\pi)^d}\, \tilde{G}_0(k)\, e^{ik(x-y)} \int \frac{d^dk'}{(2\pi)^d}\, \tilde{G}_0(k')\, e^{ik'(x-y)}.$$

Now

$$\left.\begin{array}{lll} \int d^dx & \longrightarrow & (2\pi)^d\, \delta(-k_1 - k_2 + k + k'), \\ \int d^dy & \longrightarrow & (2\pi)^d\, \delta(k + k'). \end{array}\right\}$$

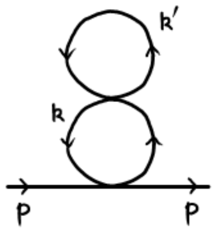
momentum conservation at each interaction vertex

$$ⓐ = 144\, g^2\, G_0(0,0) \int \frac{d^dk}{(2\pi)^d}\, \tilde{G}_0(k)^2 \int \frac{d^dp}{(2\pi)^d}\, \tilde{G}_0(p)^2\, e^{ip(x_1-x_2)}.$$

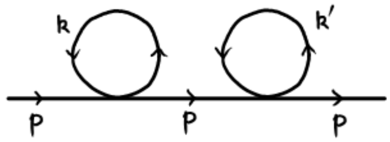
Let $\langle \phi(x)\, \phi(y) \rangle = G(x,y) = \int \frac{d^dp}{(2\pi)^d}\, \tilde{G}(p)\, e^{ip(x-y)}$ (this defines $G(x,y)$ and $\tilde{G}(p)$).

Contribution to $\tilde{G}(p)$ from

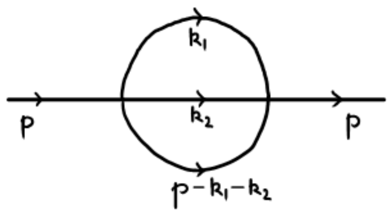
graph ⓐ $= 144\, g^2\, \tilde{G}_0(p)^2 \int \frac{d^dk}{(2\pi)^d}\, \tilde{G}_0(k)^2 \int \frac{d^dk}{(2\pi)^d}\, \tilde{G}_0(k)$.



graph ⓑ $= 144\, g^2\, \tilde{G}_0(p)^3 \left(\int \frac{d^dk}{(2\pi)^d}\, \tilde{G}_0(k) \right)^2.$



graph ⓒ $= 96\, g^2\, \tilde{G}_0(p)^2 \int \frac{d^dk_1}{(2\pi)^d} \int \frac{d^dk_2}{(2\pi)^d}\, \tilde{G}_0(k_1)\, \tilde{G}_0(k_2)\, \tilde{G}_0(p-k_1-k_2)$.



I.4 Connected Green's functions (n-point fcts).

Warm-up with toy model. Probability measure $d\mu(x)$ for $x \in \mathbb{R}$.

Moments $\quad m_n = \int x^n \, d\mu(x)\,, \qquad m_0 = 1.$

Generating function: $\quad Z(k) = \int e^{kx}\, d\mu(x) = \sum_{n=0}^{\infty} \frac{k^n}{n!} m_n\,.$

$\ln Z(k) = \sum_{n=1}^{\infty} \frac{k^n}{n!} c_n \quad$ (cumulants c_n).

Relations: $c_1 = m_1\,,\ c_2 = m_2 - m_1^2\,,\ c_3 = m_3 - 3m_2 m_1 + 2m_1^3\,, \dots$ [End warm-up].

Scalar field theory: $\quad Z[j] = \int \mathcal{D}\phi \; e^{-S[\phi] + \int d^dx\, j(x)\,\phi(x)}$.

$$\frac{\delta}{\delta j(x)} \ln Z[j]\Big|_{j=0} = \langle \phi(x) \rangle .$$

$$\frac{\delta}{\delta j(x_1)} \frac{\delta}{\delta j(x_2)} \ln Z[j]\Big|_{j=0} = \langle \phi(x_1)\,\phi(x_2) \rangle - \langle \phi(x_1)\rangle \langle \phi(x_2) \rangle .$$

$$\frac{\delta}{\delta j(x_1)} \frac{\delta}{\delta j(x_2)} \frac{\delta}{\delta j(x_3)} \ln Z[j]\Big|_{j=0} = \langle \phi(x_1)\,\phi(x_2)\,\phi(x_3) \rangle + 2\langle \phi(x_1)\rangle \langle \phi(x_2)\rangle \langle \phi(x_3)\rangle$$
$$- \langle \phi(x_1)\,\phi(x_2)\rangle \langle \phi(x_3)\rangle - \langle \phi(x_3)\,\phi(x_1)\rangle \langle \phi(x_2)\rangle - \langle \phi(x_2)\,\phi(x_3)\rangle \langle \phi(x_1)\rangle .$$

$\ln Z[j] =: F[j]$ is called the generating functional for the connected Green's functions.

Comments. Connected Green's functions (or n-point functions) correspond to n-point functions in the same way that cumulants correspond to moments. While the graphical representation of an n-point function does not contain vacuum subgraphs, it still contains graphs made from disconnected subgraphs, for example:

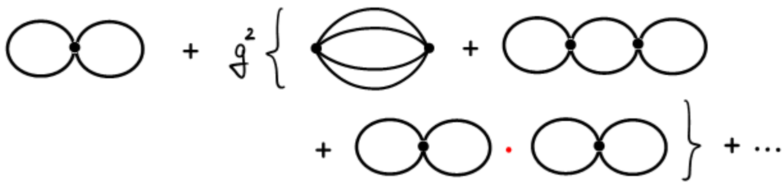
contributes to the 4-point fctn.

Such graphs cancel in the n-point functions generated by $\ln Z[j]$ (hence the name "connected").

Lecture 04

Recapitulate: $F[j] \equiv \ln Z[j] = \int \mathcal{D}\phi \; e^{-S[\phi] + \int d^dx \, j(x)\,\phi(x)}$

generating functional for connected Green's functions (n-point fcts).

Comment: $Z/Z_{free} \overset{\phi^4 \text{ theory}}{=} 1 + g \cdot$ 

$\ln Z/Z_{free} = g \cdot$ 

Note: disconnected graphs cancel in $\ln Z/Z_{free}$ ($\curvearrowleft$ linked cluster principle).

Heuristic argument: $\ln Z \propto$ free energy $\propto$ volume.

I.5 Legendre transform $\longrightarrow$ vertex functions

Recall from classical mechanics:

① Lagrangian $\mathcal{L}(v) \rightsquigarrow$ canonical momentum $\frac{\partial \mathcal{L}}{\partial v^i} = p_i$.

② $v \overset{1:1}{\longleftrightarrow} p$ ($\mathcal{L}$ convex) $\rightsquigarrow$ Hamiltonian $\mathcal{H}(p) = p_i v^i(p) - \mathcal{L}(v(p))$.

③ Legendre-T. is involutive: $\frac{\partial \mathcal{H}}{\partial p_i} = v^i$, $\mathcal{L}(v) = v^i p_i(v) - \mathcal{H}(p(v))$.

④ $\delta^i_j = \frac{\partial}{\partial p_i} p_j = \frac{\partial}{\partial p_i} \frac{\partial \mathcal{L}}{\partial v^j} = \frac{\partial v^k}{\partial p_i} \frac{\partial^2 \mathcal{L}}{\partial v^k \partial v^j} = \frac{\partial^2 \mathcal{H}}{\partial p_i \partial p_k} \frac{\partial^2 \mathcal{L}}{\partial v^k \partial v^j}$.

Transcription to field theory.

①' Put $\varphi(x) := \frac{\delta}{\delta j(x)} F[j]$. Then solve for j as a functional of φ.

②' Set $\Gamma[\varphi] = \int d^dx \, \varphi(x)\, j[\varphi](x) - F[j[\varphi]]$.

$\Gamma[\varphi]$ is called the generating functional of the "vertex functions".

③' By taking one functional derivative of the Legendre transform, one gets $\frac{\delta}{\delta \varphi(x)} \Gamma[\varphi] = j(x)$. This means that the assignment $j \mapsto \varphi$ due to $\varphi(x) = \frac{\delta}{\delta j(x)} F[j]$ is inverted by the assignment $\varphi \mapsto j$ due to $j(x) = \frac{\delta}{\delta \varphi(x)} \Gamma[\varphi]$.

④' By taking another functional derivative one obtains

$$\Gamma^{(2)}(x,y) := \frac{\delta^2}{\delta\varphi(y)\delta\varphi(x)} \Gamma[\varphi] = \frac{\delta}{\delta\varphi(y)} j(x), \text{ and}$$

$$F^{(2)}(x,y) := \frac{\delta^2}{\delta j(x)\delta j(y)} F[j] = \frac{\delta}{\delta j(x)} \varphi(y).$$

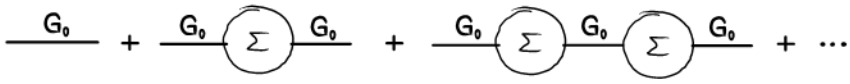
By construction — see ④ — the right-hand sides are inverse to each other (as operator kernels). Thus $\int d^d y \, \Gamma^{(2)}(x,y) \, F^{(2)}(y,z) = \delta(x-z)$.

Now let $G(x,y) = F^{(2)}(x,y)\Big|_{j=0} = \int \frac{d^d k}{(2\pi)^d} G(k) \, e^{ik(x-y)}$

and $\Upsilon(x,y) = \Gamma^{(2)}(x,y)\Big|_{\varphi=0} = \int \frac{d^d k}{(2\pi)^d} \Upsilon(k) \, e^{ik(x-y)}$.

It follows that $\Upsilon(k) = G(k)^{-1}$.

Dyson series for G:



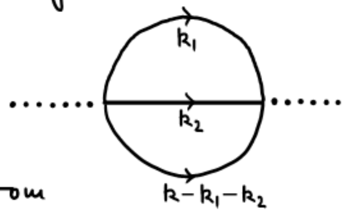
(This defines the self-energy Σ.)

Summation of the geometric series gives $G(k) = \left(G_0(k)^{-1} - \Sigma(k) \right)^{-1}$.

Therefore, $\Upsilon(k) = G(k)^{-1} = G_0(k)^{-1} - \Sigma(k)$.

Note: the graph sum for Σ contains one-particle (1P) irreducible graphs only. (These are graphs that do not become disconnected when a single line is cut.)

Example. Contribution to $\Sigma(k)$ of order g^2 in ϕ^4 theory:



Remark. A similar development (i.e. Legendre transform from $F[j]$ to $\Gamma[\varphi]$) can be made in the case of fermions and leads to an interpretation of the effective interaction as a 4-point vertex function. For algebraic consistency one takes the source field to be anti-commuting in that case.

I.6 QM & QFT on multiply connected spaces

Quantum Mechanics. Propagator:

$$K(q_f, t_f; q_i, t_i) = \sum_{h \in \pi_1(X)} R(h) \int_{[q_i \to q_f] = h} e^{iS/\hbar} \, . \qquad (\square)$$

Comments. The outer sum is over homotopy classes h (which constitute the so-called *fundamental group* $\pi_1(X)$ of position space X). The inner path integral is over paths in a given homotopy class h. The phase factor $R(h)$ is a representation $R : \pi_1(X) \longrightarrow U(1)$.

Example. Particle on a ring (S^1).

$$\pi_1(S^1) = \mathbb{Z}\,, \quad K = \sum_{n \in \mathbb{Z}} e^{in\theta} \int_{\text{windin } \# = n} e^{iS/\hbar}\,, \quad \theta = \tfrac{e}{\hbar} \iint B \ \text{(if Aharonov–Bohm geometry)}.$$

Note. Straight perturbation theory (in the sense of the present Chapter) has a problem here as it can account only for the quantum fluctuations <u>within</u> a given homotopy class. Contributions from other classes have to be added "by hand".

Quantum Field Theory.

For fields, say $\varphi : \mathbb{R}^d \cup \{\infty\} \equiv S^d \longrightarrow X$, one has an analog of $(\square)$ with a representation $R : \pi_d(X) \longrightarrow U(1)$ of the d^{th} homotopy group of the target space.

Example. Non-Abelian gauge (or Yang–Mills) theory in 4D.

$$S_{YM} = \frac{1}{g^2} \int \mathrm{Tr}\, F \wedge * F + i\theta \int \mathrm{Tr}\, F \wedge F\,,$$

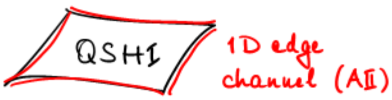
$\int \mathrm{Tr}\, F \wedge F \ \propto \ \#$ instantons, θ topological angle (causes CP violation),

$\curvearrowright$ non-perturbative effects on the Yang–Mills vacuum.

Anecdote. Phys. Rev. Lett. 69 (1992) 1584 computed the conductance $G(L)$ of thick disordered wires of any length L for the Wigner–Dyson symmetry classes A, AI, AII.

Striking result for class AII: $\frac{h}{e^2} \lim_{L \to \infty} \langle G(L) \rangle = 1/2$ (perfectly conducting channel ?!)

Kane & Mele (2005) predicted perfectly conducting edge mode for the quantum spin Hall insulator ✓



What went wrong in PRL (1992)?

Nonlinear sigma model $[0, L] \xrightarrow[\text{map}]{\text{field}} \overset{\text{supermfd}}{X} \xrightarrow[\text{proj}]{} X_0 \times X_1\,,$

$\pi_1(X_1) = \pi_1(O_4 / O_2 \times O_2) = \mathbb{Z}_2$. Microscopic analysis reveals:

$$\#\ \text{channels (thick wire)} \begin{cases} N \text{ even} \wedge R \text{ trivial} \wedge \langle G(L) \rangle \equiv g_0(L) \xrightarrow{L \to \infty} 0\,, \\ N \text{ odd} \wedge R \text{ non-trivial} \wedge \langle G(L) \rangle \equiv g_1(L) \xrightarrow{L \to \infty} 1\,. \end{cases}$$

$g_0(L)$ and $g_1(L)$ have THE SAME (!) perturbation expansion in the coupling (or L).
(This is possible as the perturbation series is asymptotic but not strongly asymptotic.)

Lecture 05

I.6 Perturbation theory for complex fermions

Recall (Wick formula for real scalar field ϕ):

$$Z/Z_{\text{free}} = e^{\frac{1}{2}\int d^dx \int d^dy \, \frac{\delta}{\delta\phi(x)} G_0(x,y) \frac{\delta}{\delta\phi(y)}} \; e^{-S_{\text{int}}[\phi]} \Big|_{\phi=0}$$

for $S = S^{(2)} + S_{\text{int}}$, $S^{(2)} = \frac{1}{2}\int d^dx \, \phi(x) \, (G_0^{-1}\cdot\phi)(x)$.

Fact (complex fermions $\psi, \bar\psi$).

Let $S = S^{(2)} + S_{\text{int}}$, $S^{(2)} = \int d^dx \, \bar\psi(x) \, (G_0^{-1}\cdot\psi)(x)$.

Then $$Z/Z_{\text{free}} = e^{\int d^dx \int d^dy \, \frac{\delta}{\delta\psi(x)} G_0(x,y) \frac{\delta}{\delta\bar\psi(y)}} \; e^{S_{\text{int}}[\bar\psi,\psi]} \Big|_{\psi=0=\bar\psi}$$

with sign convention $\int e^{\bar\psi G_0^{-1}\psi} = \operatorname{Det} G_0^{-1}$.

CHECK (signs & constants).

Choose $S = \bar\psi (G_0^{-1} + V_2)\psi$. Then $Z = \int e^S = \operatorname{Det}(G_0^{-1}+V_2)$.

On the other hand (from **Fact**),

$$Z = Z_{\text{free}} \, e^{\frac{\delta}{\delta\psi} G_0 \frac{\delta}{\delta\bar\psi}} \; e^{\bar\psi V_2 \psi} \Big|_{\psi=0=\bar\psi} = \operatorname{Det}(G_0^{-1}) \, (1 + \operatorname{Tr} G_0 V_2 + \dots)$$

$$= \operatorname{Det}(G_0^{-1}) \operatorname{Det}(1 + G_0 V_2) = \operatorname{Det}(G_0^{-1} + V_2). \checkmark$$

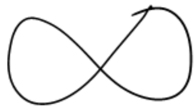
Include a source field $\zeta, \bar\zeta$ (anticommuting, for algebraic consistency):

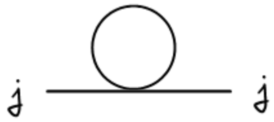
$$Z[\bar\zeta,\zeta] = \int e^{S - \int d^dx(\bar\zeta\psi - \bar\psi\zeta)}$$
$$= Z_{\text{free}} \, e^{\frac{\delta}{\delta\psi} G_0 \frac{\delta}{\delta\bar\psi}} \; e^{S_{\text{int}} - \int d^dx(\bar\zeta\psi - \bar\psi\zeta)} \Big|_{\psi=0=\bar\psi}.$$

Recall (real scalar field ϕ, $S_{int} = g \int d^dx\ \phi^4(x)$):

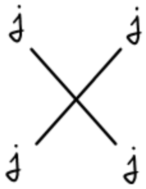
$$Z[j] = \int e^{-S + \int d^dx\, j(x)\,\phi(x)} .$$

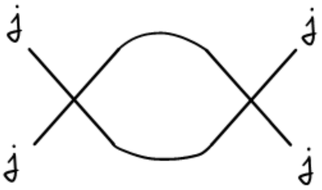
$\ln\left(Z[j] / Z_{free} \right) =$ (double expansion in coupling g and source j)

$=$  $+ \dots$ (sum over connected vacuum graphs)

$+\ j$ —— j $+$ 

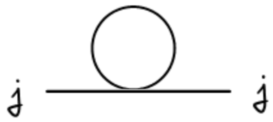
$+ \dots$

$+$ 

$+$ 

$+ \dots$

Note: j —— j $\equiv \int d^dx \int d^dy\ j(x)\, G_0(x,y)\, j(y)$,



$\equiv$ const $\cdot\, g \iiint d^dx\, d^dy\, d^dz\ j(x)\, G_0(x,z)\, G_0(z,z)\, G_0(z,y)\, j(y)$,

etc.

Another check. $e^{\frac{\delta}{\delta\psi} G_0 \frac{\delta}{\delta\bar\psi}}\ e^{-\int d^dx (\bar\zeta\psi - \bar\psi\zeta)} \Big|_{\psi=0=\bar\psi}$

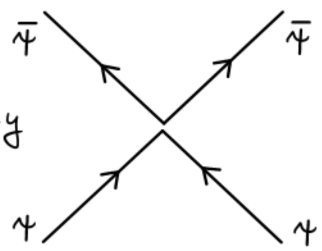
$$\equiv e^{\int d^dx \int d^dy \frac{\delta}{\delta\psi^a(x)} G_0(x,y)^a{}_b \frac{\delta}{\delta\bar\psi_b(y)}}\ e^{-\int d^dy\, \bar\psi_b(y)\zeta^b(y) + \int d^dx\, \bar\zeta_a(x)\psi^a(x)} \Big|_{\psi=0=\bar\psi}$$

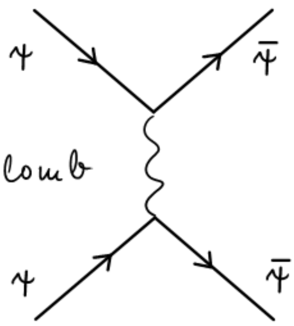
$$= 1 + \int d^dx \int d^dy \frac{\delta}{\delta\psi^a(x)} G_0(x,y)^a{}_b \left(-\zeta^b(y)\right) \int d^dx'\, \bar\zeta_{a'}(x')\psi^{a'}(x') + \dots$$

$$= 1 + \int d^dx \int d^dy\ G_0(x,y)^a{}_b \left(-\zeta^b(y)\right) \bar\zeta_a(x) + \dots = 1 + \bar\zeta G_0 \zeta + \dots = e^{\bar\zeta G_0 \zeta}.$$

Graphical notation. $\bar\zeta \longleftarrow \zeta \ = \bar\zeta G_0 \zeta = \int d^dx \int d^dy\ \bar\zeta_a(x)\, G_0(x,y)^a{}_b\, \zeta^b(y)$

(G_0 not symmetric in general)

Local two-body interaction 

Coulomb 

Some math background.

Our proof of the Wick formula uses two identities.

1 Fourier representation of the Dirac δ-distribution:

$$\delta(x-x') = \int \frac{dk}{2\pi}\, e^{ik(x-x')}.$$

For the fermionic case $\int \equiv \frac{\partial^2}{\partial\zeta\,\partial\bar\zeta}$ consider $\int e^{i\bar\zeta(\psi-\psi') - i(\bar\psi-\bar\psi')\zeta}$.

Now $\int_{\zeta,\bar\zeta} e^{i\bar\zeta(\psi-\psi') - i(\bar\psi-\bar\psi')\zeta} = (\psi-\psi')(\bar\psi-\bar\psi')$ and

$$\int_{\psi'} (\psi'-\psi)\, F(\psi') = \frac{\partial}{\partial\psi'}(\psi'-\psi)(F_0+\psi' F_1) = F_0 + \psi F_1 = F(\psi). \checkmark$$

2 Partial integration: $\int u(x)\,\frac{d}{dx} v(x)\,dx = -\int\left(\frac{d}{dx}u(x)\right) v(x)\,dx$.

Turning to the fermionic case, we recall the Berezin integral

$$\int_V F \equiv \iota(e_n)\,\iota(e_n)\cdots\iota(e_2)\,\iota(e_1)\,F \qquad (\{e_j\} \text{ basis of } V \cong \mathbb{C}^n).$$

Now $\iota(e_j)^2 = 0$ and $\iota(e_j) \equiv \frac{\partial}{\partial\psi_j}$ $\rightsquigarrow$ $0 = \int \frac{\partial}{\partial\psi_j} F$.

Hence $0 = \int \frac{\partial}{\partial\psi_j}(F_1 F_2) = \int\left(\frac{\partial}{\partial\psi_j}F_1\right)F_2 + (-1)^{\deg F_1}\int F_1 \frac{\partial}{\partial\psi_j}F_2.$ $\checkmark$

Thus both 1 and 2 are still available and the previous proof of the Wick formula goes through with some minor adjustments.

Lecture 06

I.7 QED amplitudes linear in α

Lagrangian of quantum electrodynamics:

$$\mathcal{L}_{QED} = -\frac{1}{4}\sqrt{\frac{\varepsilon_0}{\mu_0}}\, F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\left(\gamma^\mu\left(\frac{\hbar}{i}\frac{\partial}{\partial x^\mu} - eA_\mu\right) + mc\right)\psi,$$

$$S_{QED} = \int d^4x\ \mathcal{L}_{QED}.$$

Note: QED vacuum (unlike, say QCD vacuum) not so interesting.

$\rightsquigarrow$ Pass from Euclidean to Lorentzian space-time signature $(-,+,+,+)$ of $\mathbb{R}^{1+3}$ to compute real-time dynamics (scattering processes).

$$[F_{\mu\nu}F^{\mu\nu}d^4x] = \left(\frac{\text{energy}}{\text{current}}\right)^2 = \frac{\text{action}^2}{\text{charge}^2}, \quad d^4x = d^3x\ c\,dt,$$

$$\left[\sqrt{\frac{\varepsilon_0}{\mu_0}}\right] = \frac{\text{charge}^2}{\text{action}} = \frac{\text{current}}{\text{voltage}} = \text{conductance} = \frac{e^2}{\hbar}\frac{1}{2\alpha},$$

fine structure constant $\alpha = \dfrac{e^2}{4\pi\varepsilon_0\hbar c} \approx \dfrac{1}{137}$.

Develop perturbation theory using the Wick formula.

$$S_{QED} = S_{Maxwell} + S_{Dirac} + S_{int}.$$

$$e^{\frac{i}{\hbar}S_{Dirac}} = \exp\int d^4x\ \bar{\psi} D\psi, \quad D = \gamma^\mu\partial_\mu + imc/\hbar.$$

$$e^{\frac{i}{\hbar}S_{Dirac}} \overset{\text{Wick}}{\rightsquigarrow} \exp\int d^dx\int d^dy\ \frac{\delta}{\delta\psi^a(x)}(D^{-1})^a{}_b(x,y)\frac{\delta}{\delta\bar{\psi}_b(y)}$$

with Feynman propagator

$$(D^{-1})^a{}_b(x,y) = \langle \text{vac}|\,\mathcal{T}\left(\Psi^a(x)\bar{\Psi}_b(y)\right)|\text{vac}\rangle.$$

$$e^{\frac{i}{\hbar} S_{\text{Maxwell}} + \text{gauge fixing}} = \exp -\frac{i}{4\hbar}\sqrt{\frac{\varepsilon_0}{\mu_0}}\int d^4x \left(F_{\mu\nu}F^{\mu\nu} + \underbrace{2(\partial^\lambda A_\lambda)^2}_{\uparrow\ \text{Feynman gauge}}\right),$$

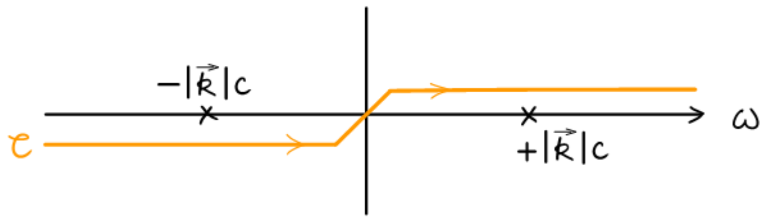
$$= \exp +\frac{i}{2\hbar}\sqrt{\frac{\varepsilon_0}{\mu_0}}\int d^4x\ A_\nu \Box A^\nu$$

with wave (or d'Alembert) operator $\Box = \partial^\mu \partial_\mu = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta$.

Wick $\rightsquigarrow$ $\exp \frac{i\hbar}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}} \int d^4x \int d^4y\ \frac{\delta}{\delta A_\nu(x)}(\Box^{-1})(x,y)\frac{\delta}{\delta A^\nu(y)}$

where $(\Box^{-1})(x,x')$ = time-ordered single-photon Green's function

$$= \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-x')}}{-k^\mu k_\mu} \equiv \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} \int_{\mathcal{C}} \frac{d\omega}{2\pi c}\frac{e^{-i\omega(t-t')}}{\omega^2/c^2 - |\vec{k}|^2}$$



Include source fields.

$$Z[j;\bar{\zeta},\zeta] = \int e^{\frac{i}{\hbar}S_{QED} + \int d^4x\,(j^\mu A_\mu + \bar{\zeta}_a \psi^a - \bar{\psi}_b \zeta^b)}$$

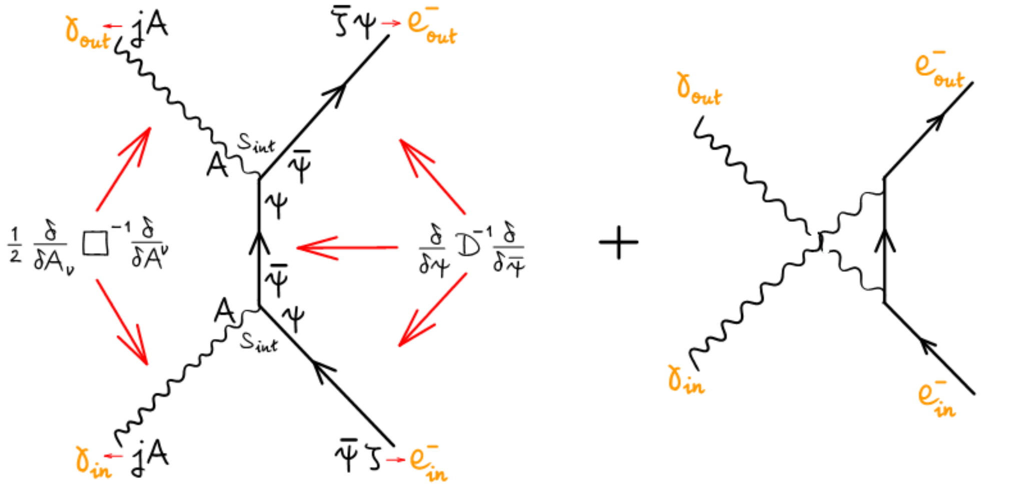
$$= Z_{\text{free}} \exp\left(\frac{\delta}{\delta\psi} D^{-1}\frac{\delta}{\delta\bar{\psi}}\right)\exp\left(\frac{1}{2}\frac{\delta}{\delta A_\nu}\Box^{-1}\frac{\delta}{\delta A^\nu}\right) e^{\frac{i}{\hbar}S_{\text{int}} + \int d^4x\,(j^\mu A_\mu + \bar{\zeta}_a\psi^a - \bar{\psi}_b\zeta^b)}\Big|_{A=\psi=\bar{\psi}=0} .$$

$$S_{\text{int}} = -\int d^4x\ A_\mu J^\mu , \qquad J^\mu = e\bar{\psi}\gamma^\mu\psi .$$

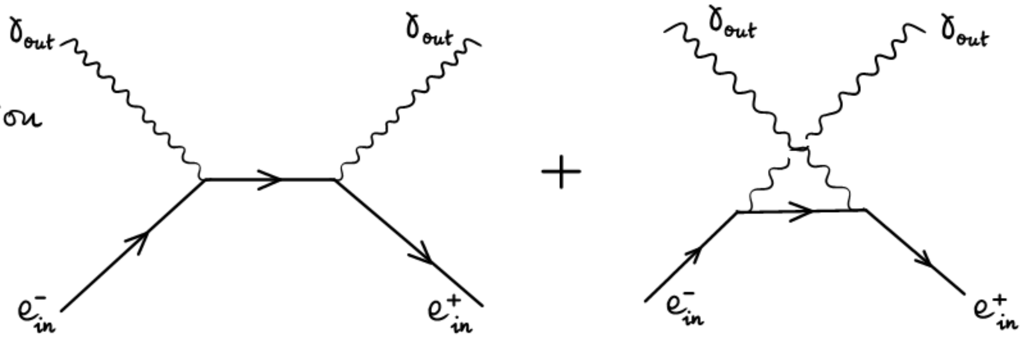
Remark. Source fields (localized not in space-time but in energy-momentum repn) can be used to project on matrix elements between incoming and outgoing scattering states.

Examples of low-order graphs.

① Compton scattering

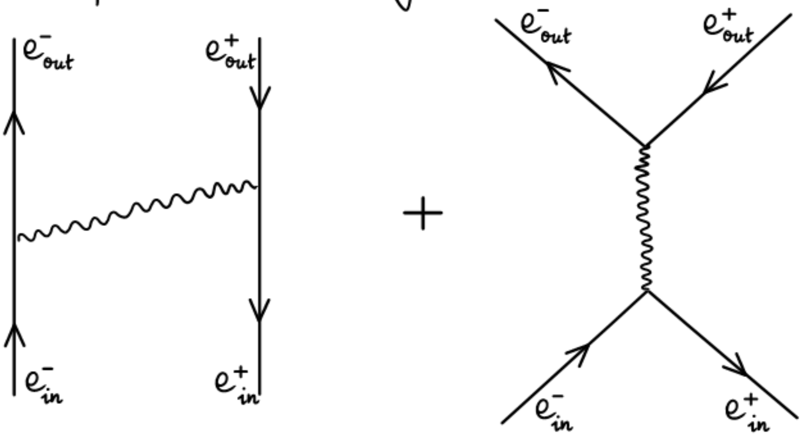


② Pair annihilation



Remark. The intermediate Feynman propagator has both electron and positron parts.

③ Electron–positron scattering



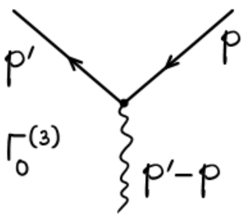
④ Electron–electron scattering (omitted).

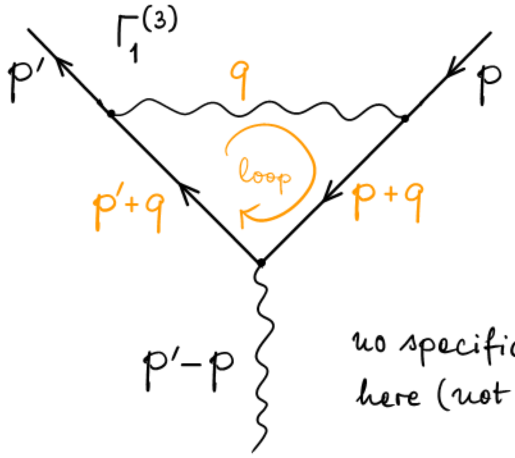
Lecture 07

The QED graphs ①–④ considered in Lecture 06 are so-called "tree graphs" (devoid of loops formed by internal lines), where the graph-internal energy-momentum variables are determined by energy-momentum conservation at the interaction vertices. Since there are no free energies/momenta to integrate over, tree graphs do not house any UV-divergencies. That changes upon turning to

One-loop graphs:

⑤ Vertex correction
= quantum correction to the bare vertex





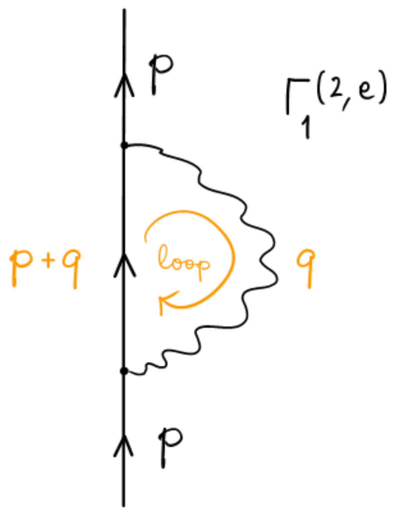
no specific time ordering intended here (not a scattering amplitude)

Remark. $\Gamma_0^{(3)} = S_{int} = -e\int d^4x\, A_\mu\, \bar{\psi}\gamma^\mu\psi$.

The integral over the loop 4-momentum q is UV-divergent $\wedge$ the 3-vertex function $\Gamma^{(3)} = \Gamma_0^{(3)} + \Gamma_1^{(3)} + \ldots$ experiences charge renormalization.

⑥ Electron self mass (or self energy).

Bare vertex: $\Gamma_0^{(2,e)} \equiv \int d^4x\, \bar{\psi} D \psi$
$= \int d^4x\, \bar{\psi}\, (\gamma^\mu\partial_\mu + imc/\hbar)\psi$.



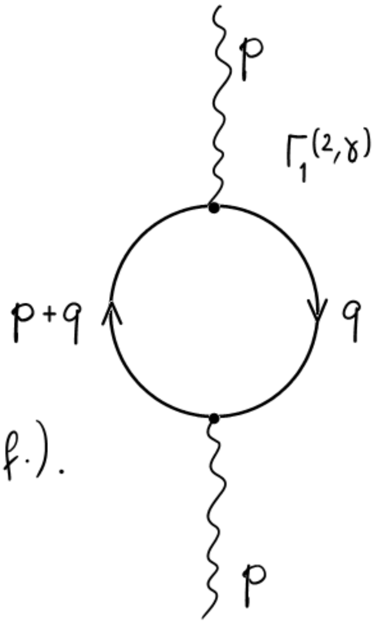
The integral over the loop 4-momentum q is again UV-divergent $\wedge$ the 2-vertex function

$\Gamma^{(2,e)} = \Gamma_0^{(2,e)} + \Gamma_1^{(2,e)} + \ldots$ experiences mass renormalization.

Note: the fermion 2-vertex function sums the Dyson series for the fermion propagator.

⑦ Vacuum polarization

= quantum correction to the inverse photon propagator



Bare vertex:

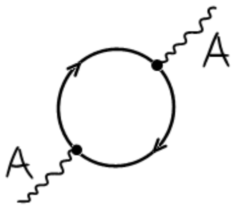
$$\Gamma_0^{(2,\gamma)} \equiv -\frac{1}{4}\sqrt{\frac{\varepsilon_0}{\mu_0}} \int d^4x \left(F_{\mu\nu} F^{\mu\nu} + g.f. \right).$$

$$\Gamma^{(2,\gamma)} = \Gamma_0^{(2,\gamma)} + \Gamma_1^{(2,\gamma)} + \dots$$

The UV-divergent correction $\Gamma_1^{(2,\gamma)}$ renormalizes the dielectric constant of the vacuum. In view of $\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}$ this can be re-interpreted as charge renormalization.

Note: the quantum-corrected photon propagator (the inverse of the 2-vertex fctn) is made from a series (Dyson) of graphs where multiple vacuum polarization insertions occur in sequence.

Computation of $\Gamma_1^{(2,\gamma)}$.



$= \Gamma_1^{(2,\gamma)} =$

$$\int d^4x \int d^4x' \frac{\delta}{\delta\psi^{a'}(x')} (D^{-1})^{a'}{}_b(x',x) \frac{\delta}{\delta\bar\psi_b(x)} \circ \int d^4x' \int d^4x \frac{\delta}{\delta\psi^a(x)} (D^{-1})^a{}_{b'}(x,x') \frac{\delta}{\delta\bar\psi_{b'}(x')}$$

$$e^2 \int d^4x\, \bar\psi_b(x) (\gamma^\mu)^b{}_a \psi^a(x) A_\mu(x) \quad \int d^4x'\, \bar\psi_{b'}(x') (\gamma^{\mu'})^{b'}{}_{a'} \psi^{a'}(x') A_{\mu'}(x') =$$

$$= - e^2 \int d^4x \int d^4x'\, A_\mu(x) (\gamma^\mu)^b{}_a (D^{-1})^a{}_{b'}(x,x') (\gamma^{\mu'})^{b'}{}_{a'} (D^{-1})^{a'}{}_b(x',x) A_{\mu'}(x')$$

in agreement with Feynman rule: (−1) for each fermion loop

$$= - e^2 \int d^4x \int d^4x'\, A_\mu(x)\, \mathrm{Tr}\left(\gamma^\mu D^{-1}(x,x') \gamma^{\mu'} D^{-1}(x',x) \right) A_{\mu'}(x').$$

Now $D = \gamma^\mu \partial_\mu + imc/\hbar \xrightarrow{F.T.} i(\not k + \lambda)$, $\not k = \gamma^\mu k_\mu$,

$\lambda = mc/\hbar$ (reduced Compton wave number).

$D^{-1}(k) = \frac{-i}{\not k + \lambda - i\varepsilon}$ (Feynman propagator in Fourier representation)

$\rightsquigarrow$ Vertex fctn $\Gamma_1^{(2,\gamma)}(k)^{\mu\mu'} = - e^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left(\gamma^\mu \frac{-i}{\not k + \lambda - i\varepsilon} \gamma^{\mu'} \frac{-i}{\not k + \not q + \lambda - i\varepsilon} \right).$

Logarithmic divergence; c.f. Exercises.

Chapter II: Symmetry Breaking & Collective Phenomena

II.1 Mean-field ground states (fermions)

(i) Hartree–Fock (number conserving) ground states

Diagonalize mean-field (one-body) Hamiltonian to find single-particle energies $\varepsilon_1, \varepsilon_2, \ldots$

Fill the n lowest-energy s.p. states to form the n-particle HF ground state:

$$|HF\rangle = c_1^\dagger c_2^\dagger \cdots c_n^\dagger |vac\rangle$$

Notation: V_h = vector space spanned by filled single-particle states ($\dim V_h = n$),

V_p = vector space spanned by empty s.-p. states,

$V_h \oplus V_p = V$ (Hilbert space for a single particle).

Remark. $|HF\rangle$ is completely determined by specifying $V_h \subset V$.

Mathematically speaking, the set of all Hartree–Fock g. states is a

Grassmann manifold $\quad \mathrm{Gr}_n(V) = \mathcal{U}(V) / \mathcal{U}(V_h^0) \times \mathcal{U}(V_p^0)$.

If $\dim V = N < \infty$ then $\dim_{\mathbb{C}} \mathrm{Gr}_n(V) = n(N-n)$.

Thouless Theorem. Fix some reference state $|HF\rangle_0$ (hence a reference decomposition $V = V_h^0 \oplus V_p^0$) with n particles. All n-particle Hartree–Fock states $|HF\rangle$ not orthogonal to $|HF\rangle_0$ can be expressed as

$$|HF\rangle_0 = \mathcal{N}^{-1/2} \exp\Big(\sum_{ph} Z_{ph}\, c_p^\dagger c_h \Big) |HF\rangle_0 \qquad \text{(generalized coherent state a la Perelomov)}$$

with complex numbers Z_{ph}.

Idea of proof. $\mathcal{U}(V)$ acts transitively on $\mathrm{Gr}_n(V)$.

Info. Another perspective (cf. QFT-1) on Hartree–Fock ground states is that a decomposition $V = V_h \oplus V_p$ uniquely determines a CAR-preserving complex structure J of $W_{\mathbb{R}}$ (the subspace of Majorana-real elements in $W = V \oplus V^*$) by $E_{+i}(J) = V_h \oplus V_p^*$ and $E_{-i}(J) = V_p \oplus V_h^*$.

Lecture 08.

(ii) Hartree–Fock–Bogoliubov (number non-conserving) ground states

Preparation: reformulate Hartree–Fock states algebraically.

$$V \equiv \mathrm{span}_{\mathbb{C}}\{c_1^\dagger, c_2^\dagger, \ldots, c_n^\dagger, \ldots\} \quad \text{creation operators,}$$
$$V^* \equiv \mathrm{span}_{\mathbb{C}}\{c_1, c_2, \ldots, c_n, \ldots\} \quad \text{annihilation operators.}$$

Let $W = V \oplus V^*$ (the space of Fock operators or field operators).

The canonical anti-commutation relations,

$$c_i^\dagger c_j^\dagger + c_j^\dagger c_i^\dagger = 0 = c_i c_j + c_j c_i, \qquad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij},$$

determine a non-degenerate symmetric bilinear form Q on W.

Recall (from QFT-1): the Clifford algebra $\mathrm{Cl}(W,Q)$ is the associative algebra of polynomials in W with relations $\psi\psi' + \psi'\psi = Q(\psi,\psi')\,\mathrm{Id}$ (for all $\psi, \psi' \in W$).

Now recall that every Hartree-Fock ground state can be viewed as a decomposition $V = V_h \oplus V_p$. Correspondingly, let $V^* = V_h^* \oplus V_p^*$. We assemble the subspaces into parts that create (W^+) resp. annihilate (W^-) excitations of the HF ground state:

$$W = V \oplus V^* = (V_p \oplus V_h^*) \oplus (V_h \oplus V_p^*) = W^+ \oplus W^-.$$

Note: $\psi|HF\rangle = 0$ for $\psi \in W^-$ and $\psi|HF\rangle \neq 0$ for $\psi \in W^+$.

Also, $Q(W^+, W^+) = 0$, $\quad Q(W^-, W^-) = 0$, $\quad Q(W^+, W^-) \neq 0$.

$\wedge$ One may identify $\mathcal{V}^* \equiv W^-$ with the dual vector space of $\mathcal{V} \equiv W^+$.

Definition: a Hartree–Fock–Bogoliubov g. state (or quasi-particle vacuum) is a choice of maximal subspace $W^- \subset W$ such that $Q(\psi,\psi') = 0$ for all $\psi,\psi' \in W^-$. (Maximal means that $W^- \simeq V^*$ has maximal dimension.)

Remarks. W^- is called the space of quasi-particle annihilation operators. The corresponding quasi-particle vacuum (or HFB state) is uniquely determined by the annihilation condition $\psi|HFB\rangle = 0$ for all $\psi \in W^-$.

The difference w.r.t. the Hartree–Fock case is that W^- need not decompose as a direct sum $V_h \oplus V_p^*$ (or equivalently, $|HFB\rangle$ need not be an eigenstate of the particle number operator). Mathematically speaking, the CAR-preserving complex structure J with eigenspaces $E_{\pm i}(J) = W^{\mp}$ need not commute with particle no.

Examples.

① $N = 1$: $V = \mathbb{C}\cdot c^\dagger$, $V^* = \mathbb{C}\cdot c$.

$W^- = \mathbb{C}\cdot\gamma$, Ansatz: $\gamma = \alpha c + \beta c^\dagger$ $(\alpha, \beta \in \mathbb{C})$.

$$0 = \tfrac{1}{2} Q(\gamma,\gamma) = \gamma^2 = (\alpha c + \beta c^\dagger)^2 = \alpha\beta\,(c c^\dagger + c^\dagger c) = \alpha\beta = 0.$$

Thus there exist but 2 possibilities ($\beta = 0$ or $\alpha = 0$):

$W^- = \mathbb{C}\cdot c \iff |HFB\rangle = |0\rangle$ (even fermion parity) OR $W^- = \mathbb{C}\cdot c^\dagger \iff |HFB\rangle = |1\rangle$ (odd fermion parity).

② $N = 2$. [Without proof, let me state the fact that the space of quasi-particle vacua is in bijection with $O(2N)/U(N)$. In the present case we have $O(4)/U(2) = S^2 \cup S^2$ (union). These 2 two-spheres will now be written down explicitly.]

Let $V^* = \mathrm{span}_{\mathbb{C}}\{c_\uparrow, c_\downarrow\}$, $V = \mathrm{span}_{\mathbb{C}}\{c_\uparrow^\dagger, c_\downarrow^\dagger\}$
and $W^- = \mathrm{span}_{\mathbb{C}}\{\gamma_\uparrow, \gamma_\downarrow\}$, $W^+ = \mathrm{span}_{\mathbb{C}}\{\gamma_\uparrow^\dagger, \gamma_\downarrow^\dagger\}$ (just notation).

Even fermion parity:

$$\begin{aligned}\gamma_\uparrow &= c_\uparrow \cos(\theta/2)\, e^{-i\phi/2} - c_\downarrow^\dagger \sin(\theta/2)\, e^{i\phi/2}\\ \gamma_\downarrow &= c_\downarrow \cos(\theta/2)\, e^{-i\phi/2} + c_\uparrow^\dagger \sin(\theta/2)\, e^{i\phi/2}\end{aligned} \iff |HFB\rangle = \cos(\theta/2)\cdot \exp\left(e^{i\phi}\tan(\theta/2)\, c_\uparrow^\dagger c_\downarrow^\dagger\right)|vac\rangle$$

where $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$ parametrize a two-sphere.

Odd fermion parity:

These are actualy number conserving (HF)

$$\begin{aligned}\gamma_\uparrow &= c_\uparrow \cos(\theta/2)\, e^{-i\phi/2} - c_\downarrow \sin(\theta/2)\, e^{i\phi/2}\\ \gamma_\downarrow &= c_\downarrow^\dagger \cos(\theta/2)\, e^{-i\phi/2} + c_\uparrow^\dagger \sin(\theta/2)\, e^{i\phi/2}\end{aligned} \iff |H\not{F}(B)\rangle = \cos(\theta/2)\cdot \exp\left(e^{i\phi}\tan(\theta/2)\, c_\uparrow^\dagger c_\downarrow\right) c_\downarrow^\dagger|vac\rangle$$

where $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$ still parametrize a two-sphere.

Info. Accepting the stated fact that θ, ϕ are spherical polar coordinates for S^2, one may wonder about the appearance of the double-valued functions $e^{\pm i\phi/2}$, $\cos(\theta/2)$, $\sin(\theta/2)$. However, that's OK: the double-valuedness reflects the fact that $\gamma_\uparrow, \gamma_\downarrow$ are local sections of a non-trivial vector bundle — that bundle is the tautological bundle which assigns to the "point" W^- in $O(2N)/U(N)$ (viewing W^- as a complex structure J of $W_{\mathbb{R}}$) the vector space W^-.

Generalization of Thouless' Theorem.

If $\langle vac|HFB\rangle \neq 0$ then there exist complex coefficients $Z_{ij} = -Z_{ji}$ such that
$$|HFB\rangle = \mathcal{N}^{-1/2} \exp\left(\tfrac{1}{2}\sum_{ij} Z_{ij}\, c_i^\dagger c_j^\dagger\right)|vac\rangle.$$

Fluctuations of particle number ?

Write $P^+ := \frac{1}{2}\sum_{ij} Z_{ij}\, c_i^\dagger c_j^\dagger$, so $|HFB\rangle \propto \exp(P^+)|vac\rangle$. If $\langle HFB|\hat{N}|HFB\rangle = N \in 2\mathbb{N}$ then $|HFB\rangle$ can be seen as an approximation to $(P^+)^{N/2}|vac\rangle$. By the law of large numbers this approximation becomes better with increasing N (Cf. the grand canonical ensemble of equilibrium statistical physics).

Info: in mathematical physics (following work by H. Araki ~ 1970) Hartree-Fock-Bogoliubov states are also known as quasi-free states.

II. 2. Mean-field theory of superconductivity

Calculating expectation values for HFB states is easy. Let $A, B, C, D \in W$ and decompose $A = A^+ + A^- \in W^+ \oplus W^-$, etc. Then

$$\langle HFB|\,ABCD\,|HFB\rangle = \langle HFB|A^-(B^+ + B^-)(C^+ + C^-)D^+|HFB\rangle$$

$$= \langle HFB|\,A^-B^+C^-D^+ + A^-B^-C^+D^+\,|HFB\rangle$$

$$= Q(A^-,B^+)\,Q(C^-,D^+) + Q(B^-,C^+)\,Q(A^-,D^+) - Q(A^-,C^+)\,Q(B^-,D^+)\,.$$

Now $Q(A^-,B^+) = \langle HFB|\,A^-B^+\,|HFB\rangle = \langle HFB|(A^+ + A^-)(B^+ + B^-)|HFB\rangle$.

To simplify the notation we write $\langle HFB|\,AB\,|HFB\rangle \equiv \langle AB\rangle_0$. Then we have the

Result: $\langle ABCD\rangle_0 = \langle AB\rangle_0\langle CD\rangle_0 + \langle AD\rangle_0\langle BC\rangle_0 - \langle AC\rangle_0\langle BD\rangle_0$.

This can be seen as a special case of the Wick principle.

We now use this result to evaluate the ground-state expectation value of a so-called pairing interaction:

$$H_{pair} = -g\int d^dx\; c_\uparrow^\dagger(x)\,c_\uparrow(x)\;c_\downarrow^\dagger(x)\,c_\downarrow(x)$$

$$= -g\int d^dx\; c_\uparrow^\dagger(x)\,c_\downarrow^\dagger(x)\;c_\downarrow(x)\,c_\uparrow(x)$$

(from short-range attraction due to deformation of the surrounding crystal).

$$\langle H_{pair}\rangle_0 \cong -g\int d^dx\;\langle c_\uparrow^\dagger(x)\,c_\downarrow^\dagger(x)\rangle_0\,\langle c_\downarrow(x)\,c_\uparrow(x)\rangle_0$$

("Cooper channel", the strongest channel in a superconductor).

Lecture 09

Recall $\langle HFB|ABCD|HFB\rangle \equiv \langle ABCD\rangle_0 = \langle AB\rangle_0\langle CD\rangle_0 + \langle AD\rangle_0\langle BC\rangle_0 - \langle AC\rangle_0\langle BD\rangle_0$.

Apply this formula to a so-called pairing interaction $H_{pair} = -g\int d^dx\; c_\uparrow^\dagger(x)\,c_\uparrow(x)\,c_\downarrow^\dagger(x)\,c_\downarrow(x)$ ⇝

$$\langle c_\uparrow^\dagger(x)c_\uparrow(x)c_\downarrow^\dagger(x)c_\downarrow(x)\rangle_0 = \underbrace{\langle c_\uparrow^\dagger(x)c_\uparrow(x)\rangle_0\langle c_\downarrow^\dagger(x)c_\downarrow(x)\rangle_0}_{\text{Hartree}} + \underbrace{\langle c_\uparrow^\dagger(x)c_\downarrow(x)\rangle_0\langle c_\uparrow(x)c_\downarrow^\dagger(x)\rangle_0}_{\text{Fock}} - \underbrace{\langle c_\uparrow^\dagger(x)c_\downarrow^\dagger(x)\rangle_0\langle c_\uparrow(x)c_\downarrow(x)\rangle_0}_{\text{Cooper}}$$

On physical grounds, keep only the Cooper channel.

We expand in Fourier modes: $c_\uparrow^\dagger(x) = \frac{1}{\sqrt{vol}}\sum_k e^{ikx}c_{k\uparrow}^\dagger$, $c_\uparrow(x) = \frac{1}{\sqrt{vol}}\sum_k e^{-ikx}c_{k\uparrow}$,

to obtain $$\langle c_\uparrow^\dagger(x)c_\downarrow^\dagger(x)\rangle_0 = \frac{1}{vol}\sum_{kk'} e^{i(k+k')x}\langle c_{k\uparrow}^\dagger c_{k'\downarrow}^\dagger\rangle_0 .$$

For a HFB state of BCS type ("Bardeen–Cooper–Schrieffer")

$$|HFB\rangle \equiv |BCS\rangle = \mathcal{N}^{-1/2}\prod_k \exp\left(z_k\, c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger\right)|vac\rangle = \prod_k\left(u_k + v_k\, c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger\right)|vac\rangle ,$$

$$v_k = \frac{z_k}{\sqrt{1+|z_k|^2}},\quad u_k = \frac{1}{\sqrt{1+|z_k|^2}},\quad \mathcal{N}^{-1/2} = \prod_k u_k ,$$

we get $$\langle c_\uparrow^\dagger(x)c_\downarrow^\dagger(x)\rangle_0 = \frac{1}{vol}\sum_k\langle c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger\rangle_0 = \frac{1}{vol}\sum_k u_k v_k = \frac{1}{vol}\sum_k\frac{\bar z_k}{1+|z_k|^2} ,$$

$$\langle c_\downarrow(x)c_\uparrow(x)\rangle_0 = \frac{1}{vol}\sum_k\frac{z_k}{1+|z_k|^2},\quad\text{and}\quad \langle c_{k\uparrow}^\dagger c_{k\uparrow}\rangle_0 = \langle c_{k\downarrow}^\dagger c_{k\downarrow}\rangle_0 = |v_k|^2 = \frac{|z_k|^2}{1+|z_k|^2} .$$

Thus the energy expectation value for electrons with single-particle energies ε_k and pairing interaction H_{pair} is $E_{HFB} \equiv E_{BCS} = 2\sum_k \varepsilon_k|v_k|^2 - \frac{g}{vol}\left|\sum_k u_k v_k\right|^2$.

To find the parameters $z_k = v_k/u_k$ of the BCS ground state (in the spirit of a variational approach), we need to minimize this energy E_{BCS} under the constraint $N = 2\sum_k |v_k|^2$ (fixed particle number). To do so, we introduce a Lagrange multiplier μ (chemical potential) and minimize the expression

$$\mathcal{E} = 2\sum_k(\varepsilon_k-\mu)|v_k|^2 - \frac{g}{vol}\left|\sum_k u_k v_k\right|^2 .$$

Introducing the quantity $\Delta := \frac{g}{vol}\sum_k u_k v_k$, we calculate the derivative

$$(1+|z_k|^2)^2\frac{\partial}{\partial\bar z_k}\mathcal{E} = 2(\varepsilon_k-\mu)z_k - \Delta + z_k^2\bar\Delta = 0 \quad\text{(at the minimum).}$$

By completing the square and then taking the (physical) square root we get

$$\bar\Delta z_k = -(\varepsilon_k-\mu) + \sqrt{(\varepsilon_k-\mu)^2+|\Delta|^2} = \Delta\bar z_k$$

and hence $$2\sqrt{(\varepsilon_k-\mu)^2+|\Delta|^2} = \Delta\bar z_k + \Delta/z_k = \Delta\,(1+|z_k|^2)/z_k .$$

We thus obtain the expression $u_k v_k = \frac{z_k}{1+|z_k|^2} = \frac{\Delta/2}{\sqrt{(\varepsilon_k-\mu)^2+|\Delta|^2}}$.

By summing over k we arrive at the so-called "gap equation"

$$\Delta = \frac{g}{\mathrm{vol}} \sum_k u_k v_k = \frac{g}{\mathrm{vol}} \sum_k \frac{\Delta/2}{\sqrt{(\varepsilon_k-\mu)^2+|\Delta|^2}} .$$

We now look for a non-trivial (approximate) solution $\Delta \neq 0$ of this equation.

Replace $\frac{1}{\mathrm{vol}} \sum_k \longrightarrow \int_{\mu-\omega_D}^{\mu+\omega_D} \nu(\varepsilon)\, d\varepsilon \approx \nu_0 \int_{\mu-\omega_D}^{\mu+\omega_D} d\varepsilon$, $\nu(\mu) \equiv \nu_0$, ω_D = cutoff.

Then the gap equation becomes

$$1 = \frac{g}{2} \int_{\mu-\omega_D}^{\mu+\omega_D} \frac{\nu(\varepsilon)\, d\varepsilon}{\sqrt{(\varepsilon-\mu)^2+|\Delta|^2}} \approx \frac{g\nu_0}{2} \int_{-\omega_D/|\Delta|}^{+\omega_D/|\Delta|} \frac{dx}{\sqrt{x^2+1}} = g\nu_0 \, \mathrm{Arsinh}\left(\omega_D/|\Delta|\right)$$

or $\frac{\omega_D}{|\Delta|} = \sinh(1/g\nu_0)$. For $g\nu_0 \gg 1$ this simplifies to $|\Delta| = 2\omega_D \, e^{-1/g\nu_0}$.

Note the non-analytic dependence on the coupling g!

Remark: As we have seen, the gap equation follows from a straight minimization. No mean-field Hamiltonian or quasi-particle energies are needed for this development. Diagonalization of the mean-field Hamiltonian is a procedure which is logically independent of the construction of the variational ground state. (It is to be done in a separate step.)

II.3 Finite-T superconductivity from field integral

Hamiltonian (as before): $H_{BCS} = \int d^dx \sum_\sigma c_\sigma^\dagger(x) \left(-\frac{\hbar^2}{2m}\nabla^2\right) c_\sigma(x) - g \int d^dx \; c_\uparrow^\dagger(x) c_\downarrow^\dagger(x) c_\downarrow(x) c_\uparrow(x)$.

Action functional for the quantum grand canonical partition function at inverse temperature $\beta = (k_B T)^{-1}$:

$$S[\bar\psi,\psi] = \int_0^\beta d\tau \left(\int d^dx \sum_\sigma \bar\psi_\sigma \left(\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu\right) \psi_\sigma - g \int d^dx \; \bar\psi_\uparrow \bar\psi_\downarrow \psi_\downarrow \psi_\uparrow \right).$$

Hubbard-Stratonovich decoupling in the Cooper channel:

$$\exp\left(g \int d\tau d^d x\, \bar\psi_\uparrow \bar\psi_\downarrow \psi_\downarrow \psi_\uparrow\right) = \int D(\bar\Delta, \Delta) \exp\left\{ -\int d\tau d^d x \left[\frac{1}{g}|\Delta|^2 - (\bar\Delta \psi_\downarrow \psi_\uparrow + \Delta \bar\psi_\uparrow \bar\psi_\downarrow) \right] \right\},$$

Auxiliary bosonic complex field: $\Delta(x,\tau) = \Delta(x,\tau+\beta)$.

Nambu spinor: $\bar\Psi = (\bar\psi_\uparrow \quad \psi_\downarrow)$, $\Psi = \begin{pmatrix} \psi_\uparrow \\ \bar\psi_\downarrow \end{pmatrix}$.

Partition function: $\mathcal{Z} = \int D(\bar\psi, \psi) \int D(\bar\Delta, \Delta) \exp\left\{ -\int d\tau d^d x \left[\frac{1}{g}|\Delta|^2 - \bar\Psi \hat{\mathcal{G}}^{-1} \Psi \right] \right\}$

Gorkov Green's function $\hat{\mathcal{G}}$: $\hat{\mathcal{G}}^{-1} = \begin{pmatrix} [\hat G_0^{(p)}]^{-1} & \Delta \\ \bar\Delta & [\hat G_0^{(h)}]^{-1} \end{pmatrix}$,

$G_0^{(p)} = \left(-\partial_\tau + \frac{\hbar^2}{2m}\nabla^2 + \mu\right)^{-1}$ ("particle"), $G_0^{(h)} = \left(-\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu\right)^{-1}$ ("hole").

Integrate out the electron field $\rightarrow$ $\mathcal{Z} = \int D(\bar{\Delta}, \Delta) \exp\left[-\frac{1}{g}\int d\tau d^d\varkappa |\Delta|^2 + \ln\det \hat{\mathcal{G}}^{-1}\right]$.

Find mean field $\Delta_0 = \text{const}$ (Cooper pair condensate) by variation with respect to $\bar{\Delta}$:

$$\frac{\Delta_0}{g} = \begin{pmatrix} -\partial_\tau + \frac{\hbar^2}{2m}\nabla^2 + \mu & \Delta_0 \\ \bar{\Delta}_0 & -\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu \end{pmatrix}^{-1}_{ph} (x,\tau; x,\tau)$$

$$= \frac{1}{L^d\beta}\sum_{k\omega} \begin{pmatrix} i\omega - \varepsilon_k + \mu & \Delta_0 \\ \bar{\Delta}_0 & i\omega + \varepsilon_k - \mu \end{pmatrix}^{-1}_{ph} = \frac{1}{L^d\beta}\sum_{k\omega} \frac{\Delta_0}{\omega^2 + (\varepsilon_k - \mu)^2 + |\Delta_0|^2} .$$

Let $(\varepsilon_k - \mu)^2 + |\Delta_0|^2 = \lambda_k^2$. Gap equation:

$$\frac{1}{g} = \frac{1}{L^d\beta}\sum_{k\omega}\frac{1}{\omega^2+\lambda_k^2} = \frac{1}{L^d}\sum_k \frac{\frac{1}{2} - n_F(\lambda_k)}{\lambda_k} \quad (n_F \text{ Fermi-Dirac distribution})$$

Use $\frac{1}{2} - n_F(\varepsilon) = \frac{1}{2} - \frac{1}{e^{\beta\varepsilon}+1} = \frac{1}{2}\frac{e^{\beta\varepsilon}-1}{e^{\beta\varepsilon}+1} = \frac{1}{2}\tanh(\beta\varepsilon/2)$.

Then $$\frac{1}{g} = \int_0^{\omega_D} d\varepsilon\; \nu(\mu+\varepsilon)\, \frac{\tanh\left(\frac{\beta}{2}\sqrt{\varepsilon^2+|\Delta_0|^2}\right)}{\sqrt{\varepsilon^2+|\Delta_0|^2}} .$$

Previous gap equation $(T=0)$ recovered by sending $\beta\to\infty$, $\tanh\to 1$.

Analysis of the gap equation. There exists a non-trivial solution $|\Delta_0| \neq 0$ for $T < T_c$ (or $\beta > \beta_c$). As the temperature approaches the critical point $(\beta\to\beta_c)$, the non-trivial solution collapses $(|\Delta_0| \to 0)$. Thus the critical point $\beta = \beta_c$ is determined by the gap equation for vanishing gap $(|\Delta_0| = 0)$:

$$\frac{1}{g\nu_0} = \int_0^{\omega_D} \frac{d\varepsilon}{\varepsilon}\tanh\left(\frac{\beta_c}{2}\varepsilon\right) = \int_0^{\beta_c\omega_D/2}\frac{dx}{x}\tanh x .$$

If $\frac{1}{g\nu_0} \gg 1$ then the main contribution to the integral must come from values of $x \gg 1$, where $\tanh x \approx 1$. Hence

$$\frac{1}{g\nu_0} = \ln(\beta_c\omega_D/2) + \text{const} \quad \text{or} \quad \beta_c^{-1} = \text{const}\cdot\omega_D\, e^{-\frac{1}{g\nu_0}}$$

(in the weak-coupling limit $g\nu_0 \ll 1$).

Exercise: $|\Delta_0| \sim \sqrt{\beta - \beta_c}$ (critical behavior).

Lecture 10

Spontaneous breaking of $U(1)$ phase rotation symmetry. Note that

$$S[\bar\psi,\psi] = \int_0^\beta d\tau \left(\int d^dx \sum_\sigma \bar\psi_\sigma \left(\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu\right)\psi_\sigma - g\int d^dx\, \bar\psi_\uparrow \bar\psi_\downarrow \psi_\downarrow \psi_\uparrow \right)$$

is invariant under $\psi_\sigma(x,\tau) \mapsto e^{-i\theta}\psi_\sigma(x,\tau)$, $\bar\psi_\sigma(x,\tau) \mapsto e^{+i\theta}\bar\psi_\sigma(x,\tau)$.

Similarly, $\mathcal{Z} = \int D(\bar\Delta,\Delta)\exp\left[-\frac{1}{g}\int d\tau d^dx\,|\Delta|^2 + \ln\det\hat{\mathcal{G}}^{-1}\right]$

is invariant under $\hat{\mathcal{G}}^{-1} \mapsto \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{+i\theta}\end{pmatrix} \hat{\mathcal{G}}^{-1} \begin{pmatrix} e^{+i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}$

or $\Delta(x,\tau) \mapsto e^{-2i\theta}\Delta(x,\tau)$. The choice of a $\Delta(x,\tau) = \Delta_0$ (with fixed phase) breaks this invariance! By Goldstone's Theorem (see a later section) it follows that there exist low-energy modes (so-called Goldstone bosons). However, in a superconductor of the type considered, there are no quasi-particle excitations of low energy. (?!)

Off-diagonal long-range order $\equiv$ ODLRO (C.N. Yang, 1962). Consider the 2-particle density matrix $\rho_2^{(N)}(X,X') := Z_N^{-1}\,\mathrm{Tr}_N\, e^{-\beta H} c^\dagger_{\sigma_1}(x_1)\, c^\dagger_{\sigma_2}(x_2)\, c_{\sigma_2'}(x_2')\, c_{\sigma_1'}(x_1')$

where the trace is over the N-particle sector of Fock space, and $X \equiv (x_1\sigma_1, x_2\sigma_2)$.

Sum rule: $\int \rho_2^{(N)}(X,X)\,dX = Z_N^{-1}\mathrm{Tr}_N\, e^{-\beta H}\hat N(\hat N - 1) = N(N-1)\,\mathrm{Tr}_N\, e^{-\beta H}/Z_N = N(N-1)$.

In the superconducting state the 2-particle density matrix acquires a macroscopically large eigenvalue: $\rho_2^{(N)}(X,X') := \lambda_{\max}\,\Psi(X)\,\bar\Psi(X') + \ldots$

where the ratio $\lambda_{\max}/N(N-1)$ stays finite in the thermodynamic limit $N\to\infty$.

For a spin-singlet s-wave superconductor the condensate wave function

$$\Psi(X) = \Psi(x_1\sigma_1, x_2\sigma_2) = -\Psi(x_2\sigma_2, x_1\sigma_1)$$

peaks at $x_1 = x_2$ and $\sigma_1 = -\sigma_2$. We observe that the amplitude

$$\Psi(x_1\sigma_1, x_2\sigma_2) \propto \langle N |\, c^\dagger_{\sigma_1}(x_1)\, c^\dagger_{\sigma_2}(x_2)\, | N-2\rangle$$

signifies a macroscopic overlap between two ground states (or equilibrium states) differing by a pair of particles. HFB mean-field theory captures this effect by a trial state which is a superposition of different particle numbers:

$$\Psi(x_1\sigma_1, x_2\sigma_2) \propto \langle \mathrm{BCS} |\, c^\dagger_{\sigma_1}(x_1)\, c^\dagger_{\sigma_2}(x_2)\, |\mathrm{BCS}\rangle .$$

II.4 Ginzburg-Landau theory

Up to now, no consideration of external fields.

Q: What happens when an electro-magnetic field is present?

Introduce electro-magnetic field (short cut!) by a local gauge transformation:

$$\psi_\sigma(x,\tau) \mapsto e^{-i\theta(x,\tau)}\psi_\sigma(x,\tau), \quad \bar\psi_\sigma(x,\tau) \mapsto e^{+i\theta(x,\tau)}\bar\psi_\sigma(x,\tau). \quad \text{This gives}$$

$$\bar\psi_\sigma \frac{\partial}{\partial\tau}\psi_\sigma \mapsto \bar\psi_\sigma\left(\frac{\partial}{\partial\tau} - i\frac{\partial\theta}{\partial\tau}\right)\psi_\sigma, \quad \bar\psi_\sigma\left(\frac{\hbar}{i}\nabla\right)^2\psi_\sigma \mapsto \bar\psi_\sigma\left(\frac{\hbar}{i}\nabla - \hbar\nabla\theta\right)^2\psi_\sigma.$$

Now $\frac{\partial\theta}{\partial\tau}$ adds to the electric potential $-e\phi$, and $\hbar\nabla\theta$ adds to the magnetic vector potential eA. Hence

$$S_{E.M.}[\bar\psi,\psi] = \int_0^\beta d\tau\left(\int d^dx \sum_\sigma \bar\psi_\sigma\left(\partial_\tau + \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - eA\right)^2 - \mu + ie\phi\right)\psi_\sigma - g\int d^dx\, \bar\psi_\uparrow\bar\psi_\downarrow\psi_\downarrow\psi_\uparrow\right).$$

By the same principle, one can introduce the electro-magnetic field in the functional $\int_0^\beta d\tau \int d^dx\, \frac{1}{g}|\Delta(x,\tau)|^2 - \mathrm{Tr}\ln\hat{\mathcal{G}}^{-1}(\Delta,\bar\Delta)$

by making a local gauge transformation $\Delta(x,\tau) \mapsto e^{-2i\theta(x,\tau)}\Delta(x,\tau)$.

In the static limit one gets the Ginzburg-Landau functional:

$$\mathcal{F} \equiv S_{\mathrm{eff,\,static}} = \int d^3x\left(\alpha|\Delta|^2 + \frac{1}{2}\beta|\Delta|^4 + \frac{1}{2m}\left|\left(\frac{\hbar}{i}\nabla - 2eA\right)\Delta\right|^2\right).$$

By variation of $\mathcal{F}$ one obtains the Ginzburg-Landau equation

$$\alpha\Delta + \beta|\Delta|^2\Delta + \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - 2eA\right)^2\Delta = 0,$$

along with the expression for the electric current density:

$$j = \frac{2e}{m}\mathrm{Re}\left(\bar\Delta\left(\frac{\hbar}{i}\nabla - 2eA\right)\Delta\right).$$

Coherence length ξ. Let $A = 0$. The substitutions $(\alpha < 0 < \beta)$

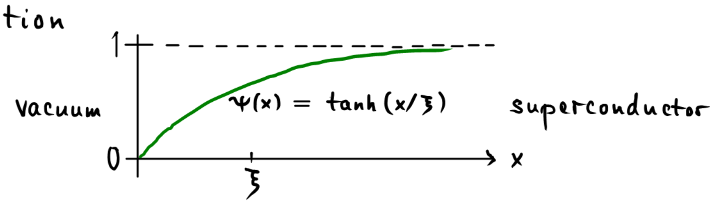
$$-\frac{\alpha}{\beta} = |\Delta_0|^2, \quad \psi = \Delta / |\Delta_0|, \quad \xi^2 = -\frac{\hbar^2}{m\alpha},$$

bring the Ginzburg–Landau equation to standard form:

$$(*) \quad \psi - |\psi|^2 \psi + \tfrac{1}{2} \xi^2 \nabla^2 \psi = 0.$$

The parameter ξ has the physical dimension of length. It sets the characteristic scale for the order parameter ψ to vary, e.g. at a superconductor–vacuum boundary. Indeed, $(*)$ has the 1D solution

$$\psi(x) = \tanh(x/\xi).$$



Penetration depth λ (Meissner effect for a type-I superconductor).

Now let $B = \operatorname{rot} A \neq 0$, but $\Delta(x) = |\Delta_0| = \text{const}$ (by choice of gauge).

Take the curl of $j = -\frac{(2e)^2}{m} |\Delta_0|^2 A$ to get $\operatorname{rot} j = -\frac{(2e)^2}{m} |\Delta_0|^2 B$.

By using $\operatorname{rot}\operatorname{rot} = \operatorname{grad}\operatorname{div} - \nabla^2$ (on vector fields), $\operatorname{div} j = 0$ (static limit) and $\operatorname{rot} B = \mu_0 j$ (Ampére) one obtains

$$\nabla^2 j = -\operatorname{rot}\operatorname{rot} j = \frac{(2e)^2}{m} |\Delta_0|^2 \operatorname{rot} B = \lambda^{-2} j \quad \text{where} \quad \lambda = \frac{\sqrt{m/\mu_0}}{2e|\Delta_0|}$$

is another parameter with the physical dimension of length. It sets the characteristic scale for the magnetic length (not) to penetrate into the superconductor. Indeed, we have $\nabla^2 B = \lambda^{-2} B$ with exponentially decreasing 1D solution $B \sim e^{-|x|/\lambda}$ ($x > 0$: superconducting region). The exponential fall off $B \sim e^{-|x|/\lambda}$ is accompanied by an exponentially decreasing screening current $j \sim e^{-|x|/\lambda}$.

Lecture 11

II.5 Goldstone's Theorem

Theme: the spontaneous breaking of continuous symmetries leads to the existence of "massless" modes. Examples: spontaneous breaking of

- rotational symmetry in a ferromagnet $\rightsquigarrow$ spin waves ("magnons");
- translational symmetry in a crystal $\rightsquigarrow$ lattice vibrations ("phonons").

Exhibit the pertinent mechanism first at the simplest example: a 2-component real field $\varphi(x) \in \mathbb{R}^2$ with SO(2)-symmetry. Action functional $S[\varphi]$ and integration measure $\mathcal{D}\varphi$ are assumed to be invariant under

$$\varphi(x) \equiv \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \mapsto g \cdot \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \quad \text{for all } g \in SO(2).$$

Consider $\langle \varphi_2(y) \rangle_h := Z^{-1} \int \mathcal{D}\varphi \; \varphi_2(y) \, e^{-S[\varphi] + h \int d^dx \, \varphi_1(x)}$.

Make the variable substitution $\varphi_a(x) \mapsto \sum_b g_{ab} \varphi_b(x)$. Then

$$\langle \varphi_2(y) \rangle_h = Z^{-1} \int \mathcal{D}\varphi \left(\sum_b g_{2b} \varphi_b(y) \right) e^{-S[\varphi] + h \int d^dx \sum_c g_{1c} \varphi_c(x)},$$

independent of $g \in SO(2)$. Let now $g = \exp(t\varepsilon)$ for $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $t \in \mathbb{R}$ and differentiate with respect to t at $t = 0$ to obtain

$$0 = Z^{-1} \int \mathcal{D}\varphi \left(\sum_b \varepsilon_{2b} \varphi_b(y) + \varphi_2(y) \, h \int d^dx \sum_c \varepsilon_{1c} \varphi_c(x) \right) e^{-S[\varphi] + h \int d^dx \, \varphi_1(x)},$$

so $\langle \varphi_1(y) \rangle_h = h \int d^dx \, \langle \varphi_2(x) \varphi_2(y) \rangle_h$.

Finally, we take the limit $h \to 0$. If the left-hand side vanishes in this limit (absence of symmetry breaking) there is no interesting consequence. However, if the limit is nonzero, then it follows that the integral $\int d^dx \, \langle \varphi_2(x) \varphi_2(y) \rangle_h$ diverges as h^{-1} for $h \to 0$. This implies the existence of a massless mode (making the correlation function long-ranged), as a mass gap in the excitation spectrum would give exponential decay $\langle \varphi_2(x) \varphi_2(y) \rangle_{h=0} \sim e^{-|x-y|/\xi}$ of the correlation function and hence a finite integral $\int d^dx \, \langle \varphi_2(x) \varphi_2(y) \rangle_{h=0} < \infty$.

Notice: it is the propagator of the transverse field components (here: φ_2) that is long-ranged.

(More) general setting: G compact Lie group. Vector space V carries G-representation $G \times V \to V$, $(g, v) \mapsto g \cdot v$. The field φ takes values $\varphi(x) \in \Gamma \subseteq V$. The symmetry group G acts on $\Gamma \subseteq V$ (by restriction).
For the integral over G with Haar measure dg we require $\int_G dg \; g \cdot \varphi(x) = 0$.
Symmetry-breaking (external) field $h \in V^*$. Pairing $\langle \, , \, \rangle : V^* \times V \to \mathbb{R}$.
G acts on V^* by the dual (linear) representation $h \mapsto g^{-1T} \cdot h$. Note $g^{-1T} \cdot 0 = 0$.
Symmetry-breaking term $= \int d^dx \, \langle h, \varphi(x) \rangle = \int d^dx \, \langle g^{-1T} \cdot h, g \cdot \varphi(x) \rangle$.

The "magnetization" $M(h) := \langle \varphi(x) \rangle_h$ is a mapping from V^* to V.
It is G-equivariant: $M(h) = g \cdot M(g^T \cdot h)$ (proof left as an exercise).

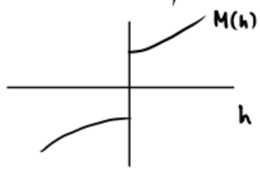
For a finite system $M(h)$ is an analytic function of h. In that case the G-symmetry cannot be broken spontaneously:

$$M(h=0) = g \cdot M(g^T \cdot 0) = \int_G dg \; g \cdot M(0) = 0 .$$

Generically, $M(h)$ vanishes linearly with h. If G acts irreducibly on V, then there exists (up to scalars) at most one (and, typically, exactly one) G-equivariant isomorphism $I : V^* \to V$ (this may be $\mathbb{C}$ anti-linear). Thus

$$M(h) = \chi \, I(h) + \ldots \quad \text{or} \quad M^a = \chi \sum I^{ab} h_b + O(h^2) .$$

$M(h=0) \neq 0$ (spontaneous magnetization and hence symmetry breaking) can only occur in a limit (such as the thermodynamic limit for an infinite system) which is non-uniform, so that the analyticity of $M(h)$ in $h = 0$ may be lost. Let $\langle h, I(h) \rangle \geq 0$ and put $\|h\| := \sqrt{\langle h, I(h) \rangle}$. Then $\|h\| = \|g \cdot h\|$, and $M(h) = \chi \, I(h) / \|h\|$ satisfies the G-equivariance condition $M(h) = g \cdot M(g^T \cdot h)$. The limit $\lim_{h \to 0} M(h)$ depends G-equivariantly on the direction $h/\|h\|$ in which $h = 0$ is approached.



Remark. The thermodynamic (or infinite-volume) limit is a mathematical idealization of real physical systems. To explain the occurrence of spontaneous symmetry-breaking in a real system, one needs to estimate time scales and argue that the time for the system to reach the G-invariant equilibrium is (much) longer than the observation time.

II.6 BEC & superfluidity (Goldstone's Thm at work)

Consider bosons with kinetic energy $H_0 = p^2/2m$ and a local repulsive two-body interaction. Use functional integral representation by a complex-valued field $\varphi(x)$ with action functional
$$S[\varphi] = \int_0^\beta d\tau \int d^dx \left(\bar\varphi \left(\partial_\tau + H_0 - \mu \right) \varphi + \tfrac{1}{2} g |\varphi|^4 \right) \quad (g>0).$$

Bose statistics: $\varphi(x,\tau) = \varphi(x,\tau+\beta) = (\beta L^d)^{-1/2} \sum_{k\omega} \varphi_{k\omega}\, e^{i(kx-\omega\tau)}$, $\omega \in \frac{2\pi}{\beta}\mathbb{Z}$.

Note: S has global $U(1)$-symmetry $\varphi(x,\tau) \mapsto e^{i\theta}\varphi(x,\tau)$, $\bar\varphi(x,\tau) \mapsto e^{-i\theta}\bar\varphi(x,\tau)$.

- Off-diagonal long-range order (ODLRO). Boson operators $a, a^\dagger$.

1-particle density matrix $\rho_1^{(N)}(x_1,x_2) := Z_N^{-1}\, \mathrm{Tr}_N\, e^{-\beta H} a^\dagger(x_1)\, a(x_2)$.

Sum rule: $\int d^dx\, \rho_1^{(N)}(x,x) = Z_N^{-1}\, \mathrm{Tr}_N\, e^{-\beta H} \hat N = N$.

Below the (Bose-Einstein) condensation temperature ρ_1 has a macroscopically large eigenvalue $\lambda_{mac} \sim N$: $\rho_1(x_1,x_2) = \lambda_{mac}\, \phi(x_1)\bar\phi(x_2) + \dots$, $\int d^dx\, |\phi(x)|^2 = 1$.

For a homogeneous system, translation invariance implies that $\phi(x) = \text{const}$. Thus in the momentum representation $\phi_k \propto \delta_{k,0}$ (macroscopic occupation of $k=0$ state).

See any basic text for Bose-Einstein condensation in the non-interacting system $(g=0)$. In the interacting system $(g>0)$ condensation requires $\mu > 0$.

Mean-field treatment: evaluate S on $\varphi(x,\tau) = \phi = \text{const}$:
$$S[\varphi(x,\tau)=\phi] = -\mu|\phi|^2 + \tfrac{1}{2} g |\phi|^4 .$$

This is minimal at $|\phi| = \sqrt{\mu/g}$. A fixed value of $\langle\varphi(x,\tau)\rangle = \phi \neq 0$ breaks the global $U(1)$-symmetry spontaneously. By Goldstone's Theorem this implies the existence of a massless mode ("Goldstone boson" $\longrightarrow$ next lecture).

Lecture 12

Recall: the spontaneous breaking of global $U(1)$ symmetry by a Bose-Einstein condensate $\langle \varphi(x,\tau)\rangle = \phi \neq 0$ implies the existence of a massless mode by Goldstone's Theorem. We are now going to exhibit the dispersion relation of that mode by direct calculation.

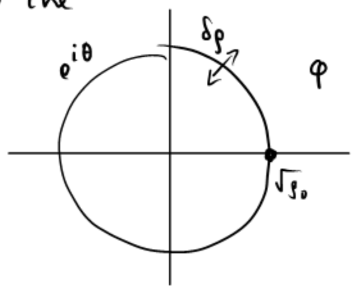
Change variables $\varphi = \sqrt{\rho}\, e^{i\theta}$, $\rho = \rho_0 + \delta\rho$, $\rho_0 = |\phi|^2$ (the Jacobian is just a constant).

Derive low-energy effective action for the U(1) degree of freedom θ by inserting $\varphi = \sqrt{\rho}\, e^{i\theta}$ into the expression for $S[\varphi]$ and expanding in $\partial_\tau\theta, \nabla\theta, \delta\rho$:

$$S[\varphi] = \int_0^\beta d\tau \int d^dx \left(i\rho\, \partial_\tau\theta + \frac{\hbar^2}{2m}\rho_0 (\nabla\theta)^2 + \frac{1}{2} g\, \delta\rho^2 \right) + \text{const}(\rho_0) + \ldots$$

Now $$\int_0^\beta d\tau \int d^dx\, i\rho_0 \partial_\tau\theta = i\int d^dx\, \rho_0 \big(\theta(\beta)-\theta(0)\big) = 2\pi i n \int d^dx\, \rho_0 = 2\pi i n N = 0 \pmod{2\pi i \mathbb{Z}}$$

and $$\frac{1}{2} g\, \delta\rho^2 + i\,\delta\rho\, \partial_\tau\theta = \frac{1}{2} g \left(\delta\rho + \frac{i}{g}\partial_\tau\theta\right)^2 + \frac{1}{2g}(\partial_\tau\theta)^2.$$ Do the Gaussian integral over the fluctuations $\delta\rho$ (with mass g)

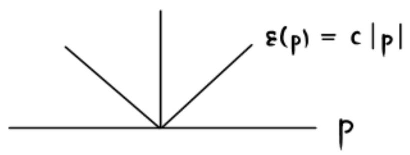


$$\rightsquigarrow\ S_{\text{eff}}[\theta] = \frac{1}{2}\int_0^\beta d\tau \int d^dx \left(\frac{1}{g}(\partial_\tau\theta)^2 + \frac{\hbar^2}{m}\rho_0 (\nabla\theta)^2 \right).$$

Starting from this effective action one proves (Fröhlich, Spencer, Simon) that spontaneous symmetry breaking <u>does</u> occur (for $d \geq 3$, or if $d=2$ for $\beta \to \infty$). One can arrange for the symmetry-broken state to be at $\theta = 0$. The fluctuations around $\theta = 0$ can be treated as a Gaussian (free) field with propagator

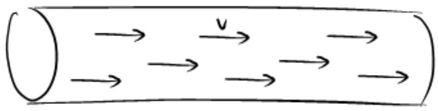
$$\left(\frac{1}{g}\omega^2 + \frac{\hbar^2}{m}\rho_0 |k|^2\right)^{-1} \propto \left(\omega^2 + c^2 |\hbar k|^2\right)^{-1}, \qquad c = \sqrt{g\frac{\rho_0}{m}} \quad \text{(speed of progation)}.$$

Switching to real time one sees that this amounts to "relativistic" excitations with dispersion $\varepsilon(p) = c\,|p|$. In particular, there is no excitation gap (✓ Goldstone's Thm).



Superfluidity. In view of the absence of an excitation gap, it is not immediately clear how to explain the observed phenomenon of superfluidity. An explanation was first given by Landau. Here is how I understand Landau's argument:

Putting the quantum fluid in motion (with constant velocity v relative to a container/pipe) we ask: what are the quasi-particle excitation energies $\varepsilon^{(v)}(p)$ of the moving fluid?



If the principle of Galilean invariance were applicable to this problem, we could say that $\varepsilon^{(v)}(p) = \varepsilon^{(v=0)}(p + mv)$, just like for a free particle with mass m. However, Galilean invariance is broken by the formation of the condensate, so a more elaborate argument is needed:

Although Galilean invariance is broken (for the quasi-particle excitations) by the actual ground state, it still applies to the underlying many-body Hamiltonian. There exist two distinguished inertial frames: the laboratory frame (= rest frame of the container) and the rest frame of the fluid. Omitting the coupling between the fluid and the container walls, the Hamiltonian of the fluid w.r.t. the latter frame is the Hamiltonian we have been working with:

$$H^{(0)} = \int d^dx \left(\frac{\hbar^2}{2m} a^\dagger(x)(-\nabla^2)\, a(x) + \frac{g}{2}\, a^\dagger(x)\, a^\dagger(x)\, a(x)\, a(x) \right).$$

The Hamiltonian $H^{(v)}$ for the fluid in motion (lab frame) then follows by a by a Galilean transformation:

$$H^{(v)} = \int d^dx \left(\frac{1}{2m} a^\dagger(x)\left(\frac{\hbar}{i}\nabla + mv\right)^2 a(x) + \frac{g}{2}\, a^\dagger(x)\, a^\dagger(x)\, a(x)\, a(x) \right) = H^{(0)} + v\cdot P + \frac{1}{2} M v^2$$

where $P = \int d^dx\, a^\dagger(x) \frac{\hbar}{i}\nabla\, a(x)$ is the operator for the total momentum and M is the total mass of the fluid.

Earlier, we diagonalized $H^{(v=0)}$ (in a way) and saw that the quasi-particle excitation energies are $\varepsilon^{(v=0)}(p) = c|p|$. We may assume that $v\cdot P$ commutes with $H^{(0)}$ and is diagonalized by the same quasi-particle basis. Thus the q.p. excitation energies of $H^{(v)}$ are $\varepsilon^{(v)}(p) = c|p| + v\cdot p \neq \varepsilon^{(v=0)}(p+mv)$.

To complete the argument, we turn on the scattering of quasi-particles off the walls of the container. As macroscopic bodies the container walls may change the (perpendicular) momentum of a quasi-particle, but they cannot absorb any energy from it. By the same token, if the fluid starts out in its ground state then it remains in it (for small velocities v), as the interaction with the walls can only create quasi-particles with excitation energy $\varepsilon^{(v)}(p) = c|p| + v \cdot p = 0$. As long as $|v| < c$, no such quasi-particle states exist.

II.7 Anderson–Higgs mechanism (overview)

Electroweak sector of the standard model of particle physics:

Fermion doublets $\psi = \begin{pmatrix} e_L \\ \nu \end{pmatrix}$ (left-handed component of electron and electron neutrino)

transform according to the fundamental representation of a $U(2)$ gauge group.

Fermion Lagrangian: $\mathcal{L}_f = \overline{\psi} \gamma^\mu (\partial_\mu + i\mathcal{A}_\mu) \psi$.

The $U(2)$ gauge field $\mathcal{A}_\mu$ takes values in the Hermitian 2×2 matrices.

(For the Lagrangian of the non-Abelian gauge field $\mathcal{A}_\mu$ see elsewhere.)

The Higgs field φ is another $U(2)$ doublet (of bosonic type): $\varphi = \begin{pmatrix} \varphi^0 \\ \varphi^1 \end{pmatrix}$.

Higgs Lagrangian: $\mathcal{L}_H = \left| (\partial_\mu + i\mathcal{A}_\mu)\varphi \right|^2 - c \left(|\varphi|^2 - |v|^2 \right)^2$.

Condensation of the Higgs field in the vacuum state:

$$\varphi_{vac} = \begin{pmatrix} 0 \\ v \end{pmatrix} \quad (\text{by the choice of basis of } \mathbb{C}^2).$$

Then
$$\left| i\mathcal{A}_\mu \varphi \right|^2 = \mathrm{Tr}\, \mathcal{A}_\mu \begin{pmatrix} 0 & 0 \\ 0 & |v|^2 \end{pmatrix} \mathcal{A}_\mu$$
$$= \mathrm{Tr} \begin{pmatrix} 0 & 0 \\ 0 & |v|^2 \end{pmatrix} \mathcal{A}_\mu^2 = 0 \cdot \underbrace{(\mathcal{A}_\mu^{00})^2}_{\substack{\text{photon} \\ \text{(massless)}}} + \underbrace{|v|^2 \mathcal{A}_\mu^{10} \mathcal{A}_\mu^{01}}_{W^\pm \text{ (massive)}} + \underbrace{|v|^2 (\mathcal{A}_\mu^{11})^2}_{Z}$$

Q: what happened to the massless Goldstone boson (in the Higgs sector) that might have been expected due to spontaneous symmetry breaking?

A (S. Coleman; facetious): "the gauge boson ate the Goldstone boson and became massive" (? $\longrightarrow$ cf. next lecture)

Lecture 13

II.8 Confusion in literature

Focus here on the Ginzburg–Landau theory of superconductivity.
(Revisit the non-Abelian Higgs model of electroweak theory later.)

- S. Weinberg ("The Quantum Theory of Fields", vol. II, p. 332):

A superconductor is simply a material in which electomagnetic gauge invariance is broken spontaneously.

Really?! Perhaps there is some sloppiness or confusion of language?

- Recall some basic definitions:

(i) Local $U(1)$ phase rotations (assume Ginzburg–Landau Th. for concreteness)

$$\Delta(x) \longrightarrow e^{2i\theta(x)}\Delta(x)$$

are symmetries for $\theta(x) = \text{const}$.

(ii) Invariance under electromagnetic $U(1)$ gauge transformations

$$\Delta(x) \longrightarrow e^{2i\theta(x)}\Delta(x), \quad A \longrightarrow A + \frac{\hbar}{e}d\theta,$$

is imposed in order to eliminate unphysical degrees of freedom.
Such transformations should not be misunderstood as "symmetries".

(iii) The transformation $\Delta(x) \longrightarrow \Delta(x), \quad A \longrightarrow A + \frac{\hbar}{e}d\theta,$
is not a gauge transformation (obviously). Rather, it is equivalent to a local $U(1)$ phase rotation by gauge invariance.

Remarks.

- Naive use of (iii) seems to be the origin of the false claim that electric charge conservation follows from electromagnetic $U(1)$ gauge invariance. By the failure of charge conservation in the mean-field approximation for superconductors, that claim might suggest that electromagnetic $U(1)$ gauge invariance is spontaneously broken.

The naive argument is $\int d^dx\, A_\mu J^\mu \overset{(iii)}{\longrightarrow} \int d^dx\, A_\mu J^\mu + \frac{\hbar}{e}\underbrace{\int d^dx\, J^\mu \partial_\mu \theta}_{\overset{p.I.}{=} -\int d^dx\, \theta\, \partial_\mu J^\mu} \underset{\text{inv}}{\overset{\text{gauge}}{\Longrightarrow}} \partial_\mu J^\mu = 0.$
The fallacy here is that (iii) is not a gauge transformation.

Remarks continued.

- As a matter of principle, electromagnetic $U(1)$ gauge invariance can never be broken spontaneously (it is in fact a necessary constraint that must be imposed in order for the theory to be consistent).

- What is broken spontaneously in a superconductor is the global $U(1)$ phase rotation symmetry $\Delta(x) \longrightarrow e^{2i\theta}\Delta(x)$.

Q: How does Ginzburg–Landau theory with spontaneously broken symmetry escape the Goldstone Theorem? (Unlike the situation with charge-neutral superfluids, cf. II.6, <u>no</u> Goldstone boson is observed in superconductors.)

A: The correct answer is somewhat subtle, as follows.

Work in the London approximation $\Delta(x) = \sqrt{n_s}\, e^{i\theta(x)}$
(n_s = constant density of superconducting condensate).

Effective action (high temperature or static limit, for now):

$$S[\theta, A] = \frac{\beta}{2} \int d^d x \left(\frac{n_s}{m} \left(\operatorname{grad}\theta - \frac{2e}{\hbar} A \right)^2 + \frac{1}{\mu_0} (\operatorname{rot} A)^2 \right).$$

Hodge decomposition: $A = A_{\text{harmonic}} + A_{\text{exact}} + A_{\text{co-exact}}$.

In our case (i.e. in Euclidean position space) $A_{\text{harmonic}} = 0$.

Laplacian (on vector fields): $\Delta = \operatorname{grad} \circ \operatorname{div} - \operatorname{rot} \circ \operatorname{rot}$.

$$A = \Delta \Delta^{-1} A = \operatorname{grad}(\Delta^{-1} \operatorname{div} A) - \operatorname{rot}(\Delta^{-1} \operatorname{rot} A) \equiv A_\parallel + A_\perp,$$

$$A_\parallel \equiv A_{\text{exact}} = \operatorname{grad}(\Delta^{-1}\operatorname{div} A), \quad A_\perp \equiv A_{\text{co-exact}} = -\operatorname{rot}(\Delta^{-1}\operatorname{rot} A).$$

Hodge-decomposed form of effective action:

$$S[\theta, A] = \frac{\beta}{2} \int d^d x \left(\frac{n_s}{m} \left(\operatorname{grad}\theta - \frac{2e}{\hbar} A_\parallel \right)^2 + \frac{n_s}{m} \left(\frac{2e}{\hbar} A_\perp \right)^2 + \frac{1}{\mu_0} (\operatorname{rot} A_\perp)^2 \right).$$

Note: in the present approximation, one may integrate out the Gaussian field θ to arrive at a reduced effective action for $A_\perp$ or $B = \mathrm{rot}\, A_\perp$:

$$S_{red}[B] = \frac{\beta}{2} \int d^dx \left(\frac{e^2 n_s}{\hbar^2 m} B \Delta^{-1} B + \frac{1}{\mu_0} B^2 \right).$$

Important: the long-range (Coulomb or 1/distance) interaction $\int d^dx\, B\Delta^{-1}B$ suppresses the magnetic field B and thus acts like a "mass term for the photon".

Textbook: "The gauge symmetry can be employed to absorb the Goldstone mode (namely, θ) into the gauge field."

Really? Physical degrees of freedom can be absorbed into unphysical d.o.f. ?!
What for? To throw them away ??

Some other texts go even further and say that the Goldstone boson θ disappears from the physics ("gets eaten up") due to its coupling to the gauge field.

Let us get this straight!
In the expression for the charge current, $j = \frac{n_s}{m}(\hbar\, \mathrm{grad}\,\theta - 2eA)$, the gauge-field component $A_\parallel$, which is usually unphysical, gets packaged with its "alter ego" $d\theta$ (as $(\mathrm{grad}\,\theta - \frac{2e}{\hbar} A_\parallel)$) and thus, by gauge invariance, becomes gauge-equivalent to the physical field $\mathrm{grad}\,\theta$!

Textbook corrected: Gauge invariance can be employed to absorb the Goldstone mode into the gauge field, thereby converting the unphysical part ($A_\parallel$) of the gauge field into a physical degree of freedom.

Counting. The current, as a vector field $j = \frac{n_s}{m}(\hbar\, \mathrm{grad}\,\theta - 2eA)$, or as a twisted 2-form $j = \frac{n_s}{m} * (\hbar\, d\theta - 2eA)$, has physical degrees of freedom given by its 3 components (in space dimension 3). In the expression given, the gauge field contributes 2 (by the 2 transverse polarizations of a photon), and θ contributes 1 more, as needed for the balance.

Our question then still remains: since the "Goldstone mode" θ is not really "eaten up" or "absorbed", how does it become non-Goldstone?

The true answer can be found by inspecting the proof of Goldstone's Theorem, which assumes locality of the interactions. Locality is important if one wants to attribute the divergence of the integrated two-point correlation function to the existence of a massless mode. Indeed, if the said divergence is just a consequence of long-range interactions, then spontaneous symmetry breaking does not imply the existence of a massless mode. In our situation with a superconductor, the supercurrents (due to phase fluctuations $d\theta$ of the would-be Golstone boson θ) do experience long-range interactions due to the electromagnetic field. This, then, is how Goldstone's Theorem is thwarted.

Reference. The whole story is covered in detail in a review article by M. Greiter (Annals of Physics, 2005).

Lecture 14

II.9 Tutorial: gauge invariance

(excerpt from lectures delivered at DPG Bad Honnef Summer School 2018; complete set of lecture notes available on my home page)

Reminder: QM for a charged particle (Schrödinger eqn)

- Gauge transformations:

$$A \mapsto A + d\chi, \quad \psi \mapsto e^{ie\chi/\hbar}\psi.$$

- Wave function ψ not gauge-invariant.
- Hamiltonian $H = \frac{1}{2m}\sum_j \left(\frac{\hbar}{i}\frac{\partial}{\partial x^j} - eA_j\right)^2$ depends on choice of gauge.

Q: Is gauge dependence inevitable?

A: No! A gauge-invariant notion of wave functions, Hamiltonians, etc., <u>does exist</u>.

> Gauge "symmetry" is a structure imposed to remove redundancy from an imperfect mathematical model of physical reality.

Dirac monopole problem: my favorite example.

Consider a charged particle moving freely in the magnetic field of a monopole with magnetic charge nh/e for $n=2$. For simplicity (and without much loss) restrict the motion to a sphere, S^2, around the monopole.

CLAIM. In this setting the wave function of the charged particle can be visualized as a vector field on S^2 (= section of the tangent bundle TS^2).

Sanity check.

Q: Shouldn't the values of a Schrödinger wave function be in $\mathbb{C}$?

A: $v(x) \in T_x S^2 \cong \mathbb{R}^2 \cong \mathbb{C}$.

Q: you mean real vector fields? (To write the Schrödinger equation, we need multiplication by $i = \sqrt{-1}$.)

A: Yes! Multiplication by i in our picture is rotation by $\pi/2$ in $T_x S^2$.

Q: What are the operators of momentum and energy?

A: Momentum $p = \frac{\hbar}{i} \nabla$ (Levi-Civita covariant derivative ∇)

Energy $= \frac{p^2}{2m}$. Note: $[\nabla_u, \nabla_v] = -i \frac{e}{\hbar} B(u,v)$.

Q: How to retrieve the picture taught in class?

A: Fix a unit-vector field $s(x)$ as a reference/standard.

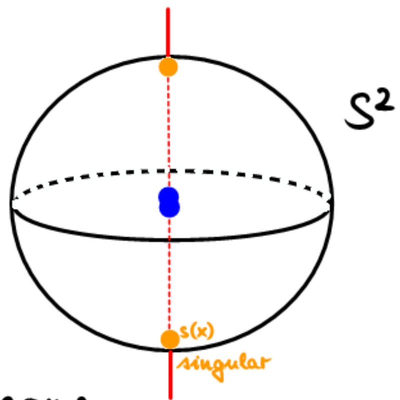
Use $T_x S^2 \longleftarrow \mathbb{C} \otimes T_x S^2$ to write $v(x) = \psi(x)\, s(x)$.

choice of gauge

$x \longmapsto \psi(x) \in \mathbb{C}$ gauge-dependent

Q: Mustn't the reference vector field $s(x)$ have some zeroes?

A: Yes, in fact $n=2$ zeroes. That's a problem for the naive approach. In the Dirac-string approach one assumes $s(x)$ with singularities. The ensuing singularities in $\psi(x)$ are attributed to fictitious magnetic flux lines entering at the singular points.



Q: This vector-field picture is great! Why isn't it used all the time?

A: In the general situation, our vector fields become sections of a complex line bundle, and working with these is not a piece of cake.

Q: What changes for monopole charge $n \neq 2$?

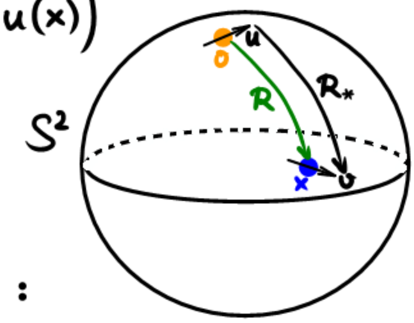
A: Write $T_x S^2 \ni v = R_* u$ where $u \in T_o S^2$ ("north pole" o)

and R_* differential of $R \in SO(3)$: $R \cdot o = x$.

Now $v(x) = R_*(x)\, u(x) = \big(R_*(x)\, g(x)\big)\big(g(x)^{-1} u(x)\big)$

with $u(x) \in T_o S^2 \cong \mathbb{C}$ gauge-dependent;

$g(x) \in SO(2) \cong U(1)$ gauge transfn.



For $n \neq 2$ form gauge equivalence classes:

$$\psi^{(n)}(x) = \big[R_*(x)\, ; u(x)\big] \equiv \big[R_*(x)\, g(x)\, ; g(x)^{-n/2} u(x)\big]$$

change the charge / or representation

Language / Notation. $S^2 = SO(3)/SO(2)$.

Associated vector bundle $E^{(n)} = SO(3) \times_{SO(2)} \mathbb{R}^2_{n/2}$

Summary. Schrödinger wave fcts are sections of a complex line bundle.

Sections s are differentiated using a connection ∇ ($\sim$ momentum $= \frac{\hbar}{i} \nabla$).

Dirac quantization condition.

electric charge $\times$ magnetic charge $/\hbar \in 2\pi\mathbb{Z}$.

Generalization.

$$\text{Associated vector bundle} \quad E = \underbrace{\overset{\text{principal bundle}}{P}}\ \times_{\underset{\text{structure group}}{G}}\ \overset{\text{standard fiber}}{V} \longrightarrow \underset{\text{base space}}{P/G}$$

Our case: $P = SO(3)$ (actually, Spin(3))

$G = SO(2)$ (actually, Spin(2))

$P/G = S^2$; $V = \mathbb{C}$ (carries G-representation)

Info. The Higgs field of the so-called "Anderson-Higgs mechanism" of electroweak theory is a section of an associated vector bundle (pulled back to spacetime) with structure group $G = SU(2) \times U(1)$ and standard fibre $V = \mathbb{C}^2$.

Final remarks. Gauge "symmetry" is <u>not</u> a symmetry!

— Associated vector bundle: $E = P \times_G V$

symmetries act here

there act the gauge transformations

— (Unitary) symmetries lead to conservation laws (Noether), but gauge "symmetries" lead to nothing of the sort.

— Symmetries can be broken (spontaneously or explicitly), but gauge "symmetries" cannot ever be broken.

Simple analogy: vector space V with basis $\{e_a\}$.

— Active transformation ($\leftsquigarrow$ physical motion):

$$v \longmapsto gv = g(e_a v^a) = (g e_b) v^b = e_a g^a{}_b v^b$$

— Passive transformation ($\leftsquigarrow$ gauge transformation):

$$v = e_a v^a = e_a (g^{-1} g)^a{}_b v^b = \tilde{e}_a g^a{}_b v^b, \quad \tilde{e}_a = e_b (g^{-1})^b{}_a$$

Reference.

— T. Tao, "What is a gauge?"
https://terrytao.wordpress.com/2008/09/27/what-is-a-gauge/

Note on "spontaneously broken gauge symmetry" (non-Abelian setting)

G-principal bundle P. Locally, $P = \underbrace{M}_{\text{spacetime}} \times \underbrace{G}_{\text{gauge group}}$

Higgs field φ is section of an associated vector bundle

$$E = P \times_G V \,, \qquad V = \mathbb{C}^2 \quad \left(\begin{array}{l}\text{Higgs doublet,} \\ \text{fundamental repn of } SU(2)\end{array}\right)$$

$$\varphi(x) = \underbrace{[p(x);\, v(x)]}_{\text{equivalence class}} = [p(x)\, g(x);\, g(x)^{-1} v(x)]$$

gauge transformation

Express the covariant derivative ∇ on the Higgs-field bundle $E \longrightarrow M$ in local basis of sections $x \mapsto s_a(x)$:

$$\nabla s_a = s_b \, \Gamma^b{}_a \quad (\text{defines the 1-forms } \Gamma^b{}_a = \Gamma^b_{\mu a}\, dx^\mu),$$

so $\nabla \varphi = \nabla(s_a \varphi^a) = (\nabla s_a)\varphi^a + s_a\, d\varphi^a = s_b\,(\delta^b{}_a d + \Gamma^b{}_a)\,\varphi^a$.

Remark. The Higgs field φ and its covariant derivative $\nabla\varphi$ are gauge-<u>invariant</u>! Gauge dependence is introduced by the transcription to the standard picture ($\longrightarrow$ particle physics), as follows.

<u>Fix</u> some gauge, i.e. pick some (local) section $x \mapsto g_0(x)$ of the G-principal bundle $P \longrightarrow M$. Use the gauge-fixing section to change the basis: $\tilde{s}_i(x) = s_a(x)\,(g_0)^a{}_i(x)$.

Then $\varphi(x) = s_a(x)\, \varphi^a(x) = \tilde{s}_i(x)\, \tilde{\varphi}^i(x)$

gauge-dependent components of Higgs field

Transcribe the covariant derivative to the gauge-dependent picture:

$$\nabla\varphi = s_b\,(\delta^b{}_a d + \Gamma^b{}_a)\,\varphi^a = \tilde{s}_j \underbrace{(g_0^{-1})^j{}_b\,(\delta^b{}_a d + \Gamma^b{}_a)\,(g_0)^a{}_i}_{= \delta^j{}_i\, d + \underbrace{(g_0^{-1}\Gamma g_0 + g_0^{-1} d g_0)^j{}_i}_{A^j{}_i}} \tilde{\varphi}^i$$

Thus $\nabla\varphi = \tilde{s}_j\, ((d + A)\,\tilde{\varphi})^j$ expresses the gauge-invariant covariant derivative $\nabla\varphi$ in terms of the gauge-dependent Higgs field $\tilde{\varphi}^i$ and the gauge-dependent non-Abelian gauge field 1-forms $A^j{}_i$.

Symmetry breaking. The group G acts on the (fibers of the) G-principal bundle P also by <u>left</u> translations $p(x) \longmapsto g(x)\,p(x)$. (Recall that gauge transformations act by <u>right</u> translations $p(x) \longmapsto p(x)\, g(x)$.) This group action is a symmetry for $g(x) = \text{const} \equiv g_0$ since $g_0 \nabla = \nabla g_0$. It is <u>this</u> symmetry (<u>not</u> the gauge "symmetry") that is spontaneously broken by the Anderson–Higgs mechanism.

Remark. Coming from a physics education, I first learned about the relevant math from a beautiful paper by Mike Stone ("Supersymmetry and the Quantum Mechanics of Spin"; Nucl. Phys. B 314 (1989) 557–586; Section 4)

Lecture 15

II.10 The H-picture of superconductivity

(H = Heisenberg, holes, heresy, Hirsch, ...)

Extracted from J.E. Hirsch: "Superconductivity begins with H"
(World Scientific, 2020)

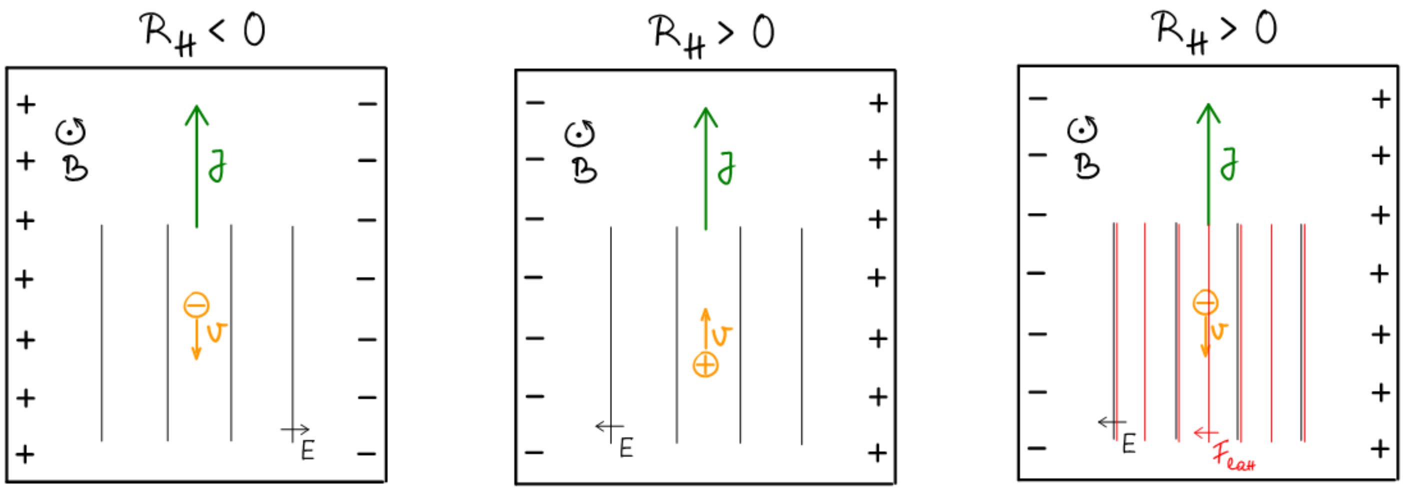
JEH: the established theory (Bardeen-Cooper-Schrieffer, 1957)

- ignores the Coulomb repulsion between electrons,
- has not predicted the T_c of any real material,
- does not explain "unconventional" superconductivity,
- is not falsifiable,
- does not explain the Meißner (-Ochsenfeld) effect.

→ Heresy: Theory of hole superconductivity (JEH).

① Hall effect (1879). Q: Why? A: ∃ strong correlation

Hall coefficient	superconductivity
$R_H < 0$ (Cu, Ag, Au)	No
$R_H > 0$ (Pb, Nb, Sn)	Yes

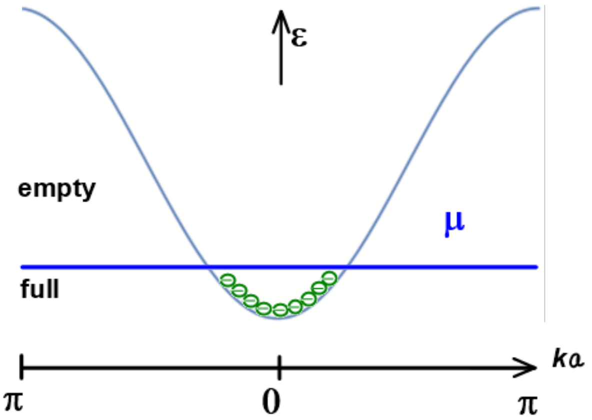


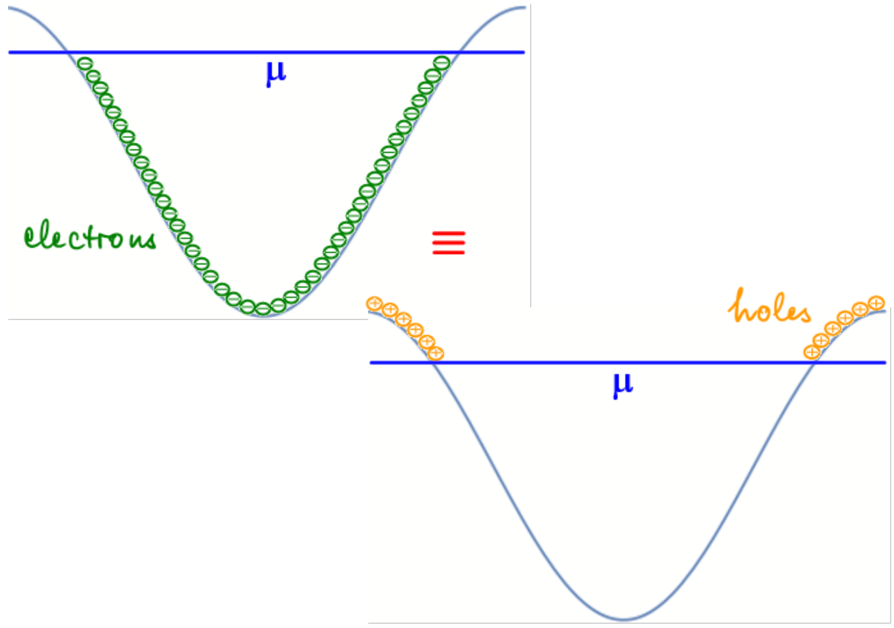
Amperian force OK (= pull on ions)

False (Ampere ↯)

Correct (Ampere OK)
$F_B + F_E + F_{latt} = 0$

② Holes (Heisenberg, 1931)

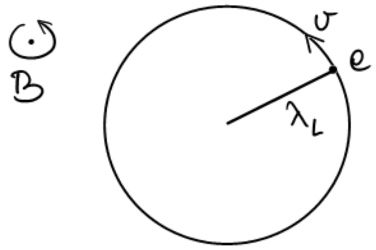




electrons	light	fast	decouple from ions	smooth wfctn	bonding
holes	heavy	slow	deform lattice	spiky wfctn	antibonding

Note: analogy with electrons vs. positrons (= holes) misleading !

③ Slater orbits (1937)



$$|e\, v\, B| = m \frac{|v|^2}{\lambda_L}$$

$$\rightsquigarrow \quad |v| = \lambda_L \frac{|eB|}{m} = \lambda_L \mu_0 \frac{|e|}{m} |H| .$$

magnetic moment / orbit $= \pi \lambda_L^2 \cdot |e| \frac{|v|}{2\pi\lambda_L} = \frac{e^2}{2m} \lambda_L^2 \mu_0 |H|$

$\rightsquigarrow$ magnetic excitation (for orbit density n_s) $= \frac{e^2}{2m} \lambda_L^2 n_s \mu_0 |H| \equiv |H|$

perfect diamagnet

$\rightsquigarrow \quad \lambda_L = \sqrt{\frac{2m}{e^2 n_s \mu_0}}$ (~ London penetration depth).

Slater: $\lambda_L = 137\, a_0 \approx a_0 / \alpha$

(Bohr radius a_0, fine structure constant α; assume $n_s = (2a_0)^{-3}$).

④ Mesoscopic orbits from spin-orbit coupling

IDEA: ferromagnetism from pre-existing spin magnetic moments —
$\sim$ superconductivity from pre-existing orbital magnetic moments ?

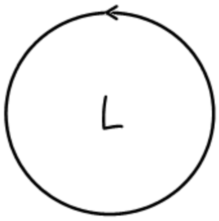
By the theory of special relativity, a spin magnetic moment μ_{magn} in motion with velocity v acquires an electric dipole moment μ_{elec} given by the formula $\mu^j_{elec} = (\mu_{magn})^j{}_\ell \, v^\ell / c^2$.

$\rightsquigarrow$ spin-orbit energy $= - E_j \, \mu^j_{elec} = - E_j (\mu_{magn})^j{}_\ell \, v^\ell / c^2$.

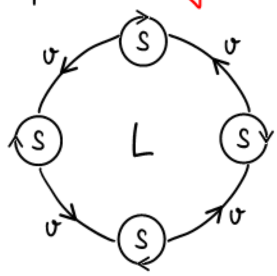
In the case of electrons: $(\mu_{magn})^j{}_\ell = - \dfrac{|e|\hbar}{2im} [\sigma^j, \sigma_\ell]$ (Pauli matrices $\sigma_\ell \equiv \sigma^\ell$).

Check: $(\mu_m)^x{}_y \equiv \mu^z_m = - \dfrac{|e|\hbar}{m} \sigma^z$ ✓

Now imagine placing (superfluid) electrons in a quantum state of total angular momentum zero:



$+ \; S = \frac{\hbar}{2} - \frac{\hbar}{2} = 0$. That is,



Q: What is the (orbit) radius λ_L of such a motion? A: Balance the forces:

Typical electric field gradient (on dimensional grounds) $\sim \dfrac{|e| \, n_s}{\varepsilon_0}$.

$$\rightsquigarrow \text{spin-orbit force} = \frac{|e| \, n_s}{\varepsilon_0} \, \frac{|e|\hbar}{m} \, \frac{|v|}{c^2} = \frac{e^2 n_s \mu_0}{m} \hbar |v| .$$

$$\text{Centrifugal force} = m \frac{|v|^2}{\lambda_L} = (m |v| \lambda_L) \frac{|v|}{\lambda_L^2} \equiv \frac{\hbar}{2} \frac{|v|}{\lambda_L^2} .$$

Force balance $\Longrightarrow \dfrac{1}{2\lambda_L^2} = \dfrac{e^2 n_s \mu_0}{m}$. Thus the spin-orbit coupling has the correct magnitude to make for mesoscopic orbits of Slater size.

Remarks.

1. Both (electron) spin states need to be considered: charge currents cancel but spin currents add up (note: the existence of spin currents is <u>not</u> in conflict with any symmetries. Recall anecdote: momentum current $\leftarrow$ KPK).

2. In order for the spin-orbit scenario to take place, the superconductor must be electrically polarized! (Superconductor as a "giant atom")

Lecture 16.

Discussion. Superconductor electrically polarized ? (impossible for metals !)

— superconductor is in a macroscopic quantum state minimizing the <u>sum</u> of potential and kinetic energy (analogy with hydrogen atom).

— The wave function of the electrons expands more than that of the ions (of course). The effect is large in the "hole"-type situation with many mobile electrons of high (initial) energy.

— From the excess of electric charge at the surface of a superconductor one expects (Hirsch, 1989) an asymmetry in the tunneling conductance. Such an asymmetry is seen in experiments on the cuprate superconductors (1998).

4 Charge expulsion ($\rightsquigarrow$ Meißner effect)

hydrodynamics: velocity vector field v;

Lie derivative $\mathcal{L}_v = d \circ \iota(v) + \iota(v) \circ d$.

Example: $\frac{d}{dt}\rho = \frac{\partial}{\partial t}\rho + \mathcal{L}_v \rho = \dot{\rho} + dj = 0 \qquad (j = \iota(v)\rho)$.

Momentum one-form $p = p_i \, dx^i = m \langle v, \cdot \rangle$.

$$\frac{d}{dt} p \overset{\text{Newton}}{=} e\,(E - \iota(v) B)$$

$$\rightsquigarrow \frac{d}{dt} dp = e\,(dE - d\iota(v) B) \overset{\text{Faraday}}{=} e\,(-\dot{B} - \mathcal{L}_v B) = -e \frac{d}{dt} B .$$

[**Check:** $\text{E.M.F} = \frac{1}{e} \oint_{\partial\Sigma} \frac{d}{dt} p \overset{\text{Stokes}}{=} -\iint_{\Sigma} \frac{d}{dt} B =$ rate of changing flux.]

Let $\omega = dp + eB$ ("generalized vorticity"). Then we have the conservation law $\frac{d}{dt}\omega = 0$ or $\frac{\partial}{\partial t}\omega = -\mathcal{L}_v \omega$.

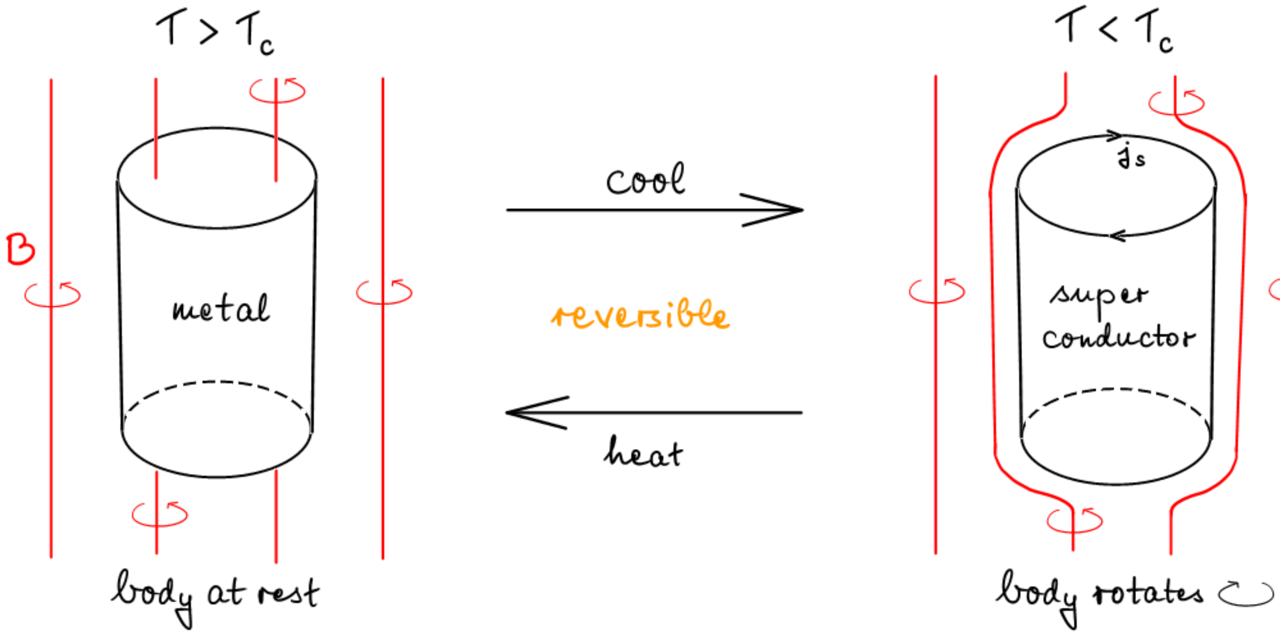
Second London equation: $\omega = 0$ (in the stationary state of a superconductor).

Indeed, $\omega = dp + eB = d(\hbar\, d\theta - eA) + eB = \hbar\, d^2\theta - eB + eB = 0$.

Note: to reach the superconducting state ($\omega = 0$), radial flow is needed !

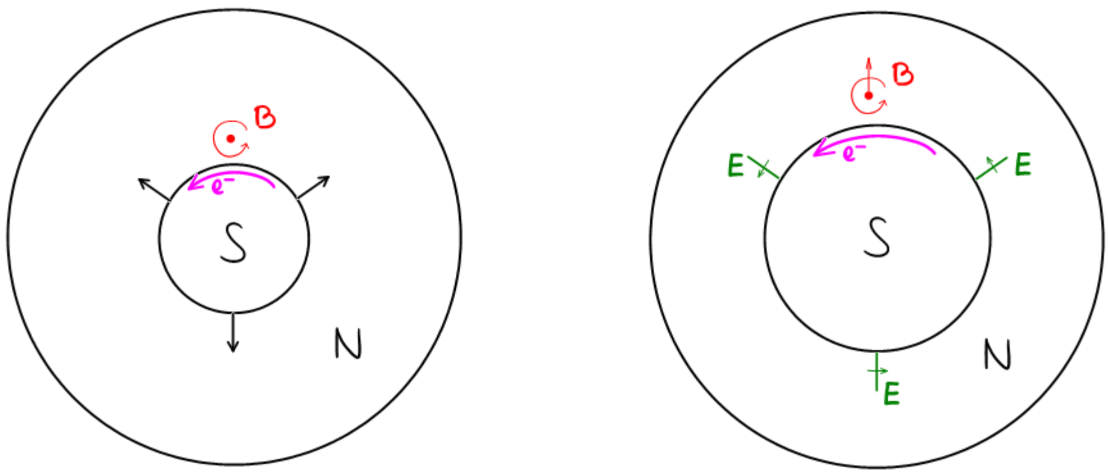
K.M. Koch (1940): outward heat flow & thermoelectric effect.

⑤ Meißner effect explained.



Remark. The big challenge is to explain how to go from left to right.

Top view of the dynamics: the superconducting condensate appears first in the center and expands radially outward:



The panel on the right shows the dynamical situation at a later time, but also displays the Faraday electric field due to the outward motion of the magnetic flux lines.

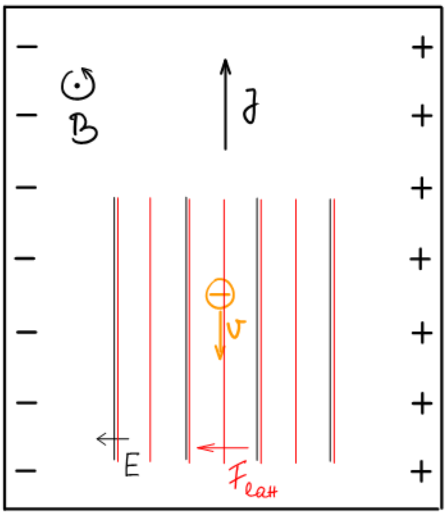
As the superfluid electrons expand radially outward, they pick up azimuthal speed by the action of the magnetic Lorentz force. (The ones that stay behind get stopped by the Faraday field.) By the law of conservation of angular momentum, the superconducting body must start rotating clockwise. Where does the torque come from? (The Faraday field pulls the positively charged ions in the counterclockwise direction!)

Resolution of puzzle. The idea is very similar to that for the Hall effect of metals with positive Hall coefficient.

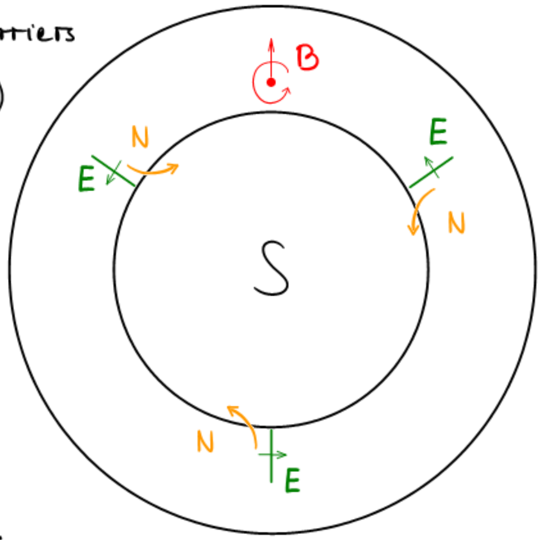
Reminder Hall effect ($R_H > 0$):

$$F_B + F_E + F_{latt} = 0$$

$$\rightarrow + \rightarrow + \longleftarrow = 0$$



To avoid inordinate pile-up of electrical charge due to the superfluid electrons moving radially outward, there is backflow of normal-fluid electrons moving radially inward. It is crucial that the inflowing electrons are charge carriers of the hole-type situation (conduction band almost full) with strong coupling to the ion lattice: both the magnetic Lorentz force and the electric Faraday force pull the inflowing electrons in the clockwise direction, and by the electrons' hole characteristics that pull gets transferred to the lattice of ions (overpowering the pull of the Faraday force in the opposite direction).



Synopsis (hole superconductivity).

- In the condensation to the superconducting state, "mesoscopic atoms" of size $\lambda_{London} \sim 137\, a_{Bohr}$ are formed. (Spin-orbit coupling plays a role there.)
- The orbit expansion ($a_{Bohr} \longrightarrow \lambda_{London}$) leads to a macroscopic quantum state with lower energy by reducing the quantum kinetic energy.
- The superconductor is electrically polarized and carries spin currents at the surface (even with no magnetic field present).
- Meißner effect: outflowing superfluid electrons acquire azimuthal speed to screen the magnetic field; inflowing normal electrons (hole type) transfer angular momentum to the solid body.
- The "hole mechanism" is proposed to be universal for superconductivity!

Lecture 17.

Chapter III : Renormalization (an introduction)

[The ideas here are fairly simple, the exact calculations are not ...]

III.1 Some historical perspective

(i) View from particle physics (pre-Wilson era, 1950's and 1960's)

Quantum field theory suffers from the appearance of ultraviolet (UV) divergences. Regularization is needed to cut off the infinite contributions from short wavelength or high-energy modes. By the introduction of a cutoff, physical observables become cutoff-dependent and hence unpredictable (in the first instance). That looks like a disaster! How to repair the problem and recover predictability?

Recipe: Add so-called "counter terms" to the (bare) Lagrangian. Fine-tune the parameters of the counter terms in such a way as to cancel the cutoff dependence. Carry out this "program with counter terms" order by order in perturbation theory (uff!). Two distinct scenarios emerge.

Scenario 1: there is a proliferation of counter terms, i.e. with increasing order of the perturbation expansion more and more (new types of) counter terms must be introduced to cancel the unwanted cutoff dependence. (This is what happens when one applies standard quantization to Einstein classical gravity.) The situation then is hopeless (no predictive power). One says that the theory is non-renormalizable.

Scenario 2. A finite number (say, r) of counter terms suffices to cancel the cutoff dependence in all orders of perturbation theory. Such a theory is called r-parameter renormalizable. It is predictive once r unknown parameters have been determined by matching to known observables.

Example. Quantum electrodynamics in 3+1 dimensions is 3-parameter renormalizable. Thus in QED one can make do with 3 types of counter term. Each of these is already present in the bare Lagrangian. The unknown parameters are field normalization $(\sqrt{\varepsilon_0/\mu_0})$ bare electron mass (m), and bare electron charge (e), all defined at some arbitrary UV cutoff scale. The 3 counter terms serve to cancel the 3 basic one-loop divergences of vacuum polarization, electron self mass, and vertex correction (cf. Chapter I, Section 7), which appear as (the sole) UV-divergent building blocks in higher-order P.T. graphs.

Remark. The scenario of renormalizability was a major guide in the formulation of the electro-weak theory (Glashow–Salam–Weinberg) and quantum chromodynamics (Gell-Mann, Fritzsch). A rough criterion for renormalizability is that the coupling parameters of the theory be dimensionless (true for nonlinear sigma models in 2D and non-Abelian gauge theories in 4D).

ii Wilson picture of renormalization (inspired by condensed matter physics). Nowadays one thinks about renormalization in a different way (especially, outside of the particle physics community), motivated by physical systems that come with a natural UV cutoff (nixing the worry about dependence on an arbitrary choice of cutoff) and where the interesting observations are made in the infrared (i.e. at long wavelengths), not in the ultraviolet. The change of thinking was brought about by work of Kenneth G. Wilson published in the early 1970's (rewarded by Nobel Prize for Physics in 1982). In Wilson's picture the focus shifts to effective field theories (given by effective actions), and non-renormalizability becomes less of an issue.

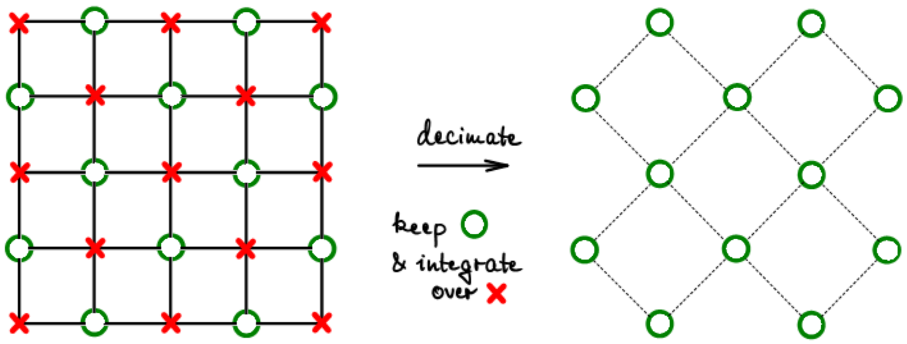
Roughly speaking, Wilson's strategy is to do the functional integral (or statistical sum) <u>sequentially</u>, starting with the large wavevector (or high frequency) modes and proceeding in the direction of small wave vector (or low frequency) modes. The sequential process drives a so-called RG flow (renormalization group) of the action functional.

II.2 Decimation

The Wilson strategy can be implemented in the momentum or in the position representation. In the latter case one speaks of real-space renormalization.

The conceptually simplest real-space RG scheme is decimation (assuming a lattice discretization of the field theory). To convey the idea we consider

Example. Square lattice, with field $\phi : \mathbb{Z}^2 \longrightarrow M$ (e.g. $M = \mathbb{R}$), partition function $Z = \int \mathcal{D}\phi \, e^{-S[\phi]}$.



$$Z = \int \mathcal{D}\phi_{\circ} \int \mathcal{D}\phi_{\times} e^{-S_{bare}[\phi_{\circ} + \phi_{\times}]} \equiv \int \mathcal{D}\phi_{\circ} \, e^{-S_{eff}[\phi_{\circ}]} .$$

Remarks. 1. The decimation scheme can be iterated to produce a recursion

$$S_{bare} \longrightarrow S_{eff}^{(1)} \longrightarrow S_{eff}^{(2)} \longrightarrow \cdots$$

2. Caveat: starting with local interactions for S_{bare}, it is not guaranteed that the interactions will still be local for $S_{eff}^{(1)}$, $S_{eff}^{(2)}$, etc. (More interactions with new parameters may keep appearing and the scheme may become difficult to control unless some approximation of truncation is made.)

3. Decimation in 1D does preserve locality of the interactions.

Example (for later use): 1D Ising model.

Ising spins $s \in \{\pm 1\}$. Energy $H = -J \sum_{n \in \mathbb{Z}} s_n s_{n+1}$ $(J > 0)$.

$$Z = \sum e^{-\beta H} = \sum_{\{s_n\}} e^{K \sum_{n \in \mathbb{Z}} s_n s_{n+1}} \quad (K = \beta J).$$ Decimation step:

Sum over spin $s_n \in \{\pm 1\}$: $\sum_{s_n} e^{K s_{n-1} s_n} e^{K s_n s_{n+1}} = 2 \cosh\left(K(s_{n-1} + s_{n+1})\right)$.

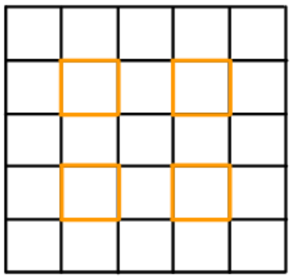
Renormalization of coupling $K \longmapsto K'$:

$$\frac{\text{Weight}(s_{n-1} = +s_{n+1})}{\text{Weight}(s_{n-1} = -s_{n+1})} = \cosh(2K) \equiv e^{2K'} \rightsquigarrow K' = \frac{1}{2} \ln \cosh(2K) \equiv f(K).$$

Note: there is only one fixed point $(K_* = f(K_*) = 0)$ $\rightsquigarrow$ 1D Ising model always in high-T phase.

III.3 Kadanoff block spin transformation

To illustrate the idea, consider a real scalar field $\phi : \mathbb{Z}^2 \longrightarrow \mathbb{R}$. (More generally, the target space could be any Abelian group.)



Organisation by blocks: every site i of the original lattice $\mathbb{Z}^2$ belongs to exactly one block $\boxed{b}$.
For each block b, introduce a new variable, $\Phi_b \in \mathbb{R}$.

Insert $1 = \int d\Phi_b \, \delta\left(\Phi_b - \sum_{i\in b} \phi_i\right)$ under the functional integral and interchange the order of integration:

$$Z = \int \mathcal{D}\phi \, e^{-S_{bare}[\phi]} = \int \mathcal{D}\phi \, e^{-S_{bare}[\phi]} \int \mathcal{D}\Phi \prod_{\text{blocks } b} \delta\left(\Phi_b - \sum_{i\in b} \phi_i\right)$$

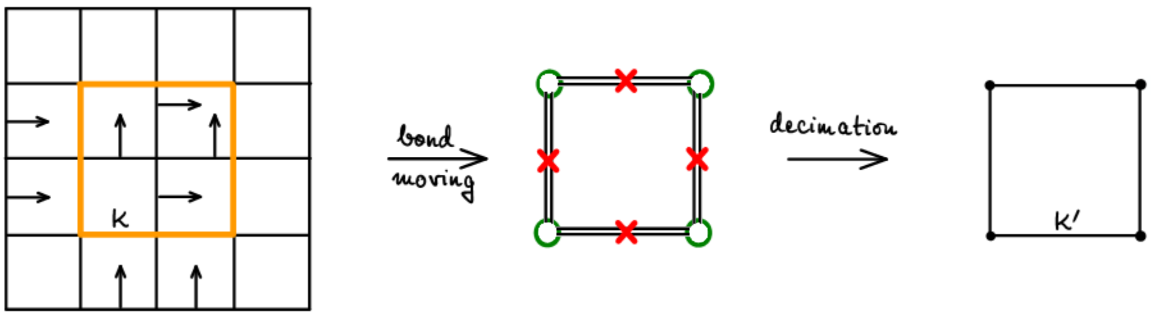
$$= \int \mathcal{D}\Phi \int \mathcal{D}\phi \prod_b \delta\left(\Phi_b - \sum_{i\in b}\phi_i\right) e^{-S_{bare}[\phi]} = \int \mathcal{D}\Phi \, e^{-S^{(1)}_{eff}[\Phi]}.$$

Remarks.

- In probability theory, this would be called push forward (of measure or statistical weight) by the map: sites $\longrightarrow$ blocks.
- Both energy and entropy play a role here (of course).
- Iteration gives RG flow $S_{bare} \longrightarrow S^{(1)}_{eff} \longrightarrow S^{(2)}_{eff} \longrightarrow \ldots$
- The method comes with some flexibility: the δ-function can be replaced by some other, smooth function (of total mass $= 1$).

III.4 Migdal–Kadanoff approximation

Decimation and Kadanoff block spin transformation are exact steps; as such they are typically difficult to implement as a recursive scheme to be iterated again and again. Now we will meet a scheme that is approximate (but still reasonable) and can be implemented with relative ease. We illustrate the idea again at the example of the 2D Ising model:



Migdal–Kadanoff RG scheme:

$$\text{coupling } K \xrightarrow{\text{move bonds}} 2K \xrightarrow{\text{decimate}} \frac{1}{2}\ln\cosh(4K) \equiv K'.$$

Fixed points. $K_* = f(K_*)$, $f(K) = \frac{1}{2}\ln\cosh(4K)$.

- $K_* = 0$ (infinite temperature; paramagnetic phase; cf. 1D Ising model)
- $K_* = \infty$ (zero temperature; ferromagnetic phase; SSB)
- $K_* \notin \{0,\infty\}$ (critical temperature; phase transition).

Info. The Migdal–Kadanoff approximation correctly predicts that the one-loop RG beta function for nonlinear sigma models in two dimensions is given by the Ricci curvature of the target space.

Lecture 18.

Perspective on MK RG scheme:

- A. Migdal (JETP, 1975) pointed out that "asymptotic freedom" (discovered in 1973 by Gross & Wilczek; Politzer) in 4D non-Abelian gauge theories can be obtained (with good accuracy) from an approximate recursive RG scheme.
- L. Kadanoff (Ann. Phys., 1976) put Migdal's recursion in the perspective of earlier work on real-space renormalization and commented on its validity.

Info ($\to$ Exercise Sheet 9).

Classical Heisenberg spin model $\xrightarrow[\text{2D, low } T]{\text{continuum limit}}$ $O(3)$ nonlinear σ model

$O(3)$ NLσM : field (spacetime with Euclidean signature) $n : \mathbb{R}^2 \longrightarrow S^2 \subset \mathbb{R}^3$,

$$n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, \quad n^2 = 1, \quad H = J \int_{\mathbb{R}^2} d^2x \, (\nabla n)^2, \quad Z = \int \mathcal{D}n \; e^{-K \int d^2x \, (\nabla n)^2}, \quad K = \beta J .$$

In perturbation theory (valid for large K) one finds (e.g. by background-field renormalization, see later) that K decreases when the short-distance cutoff a is increased. (This is the 2D NLσM analog of asymptotic freedom for 4D gauge theories.)

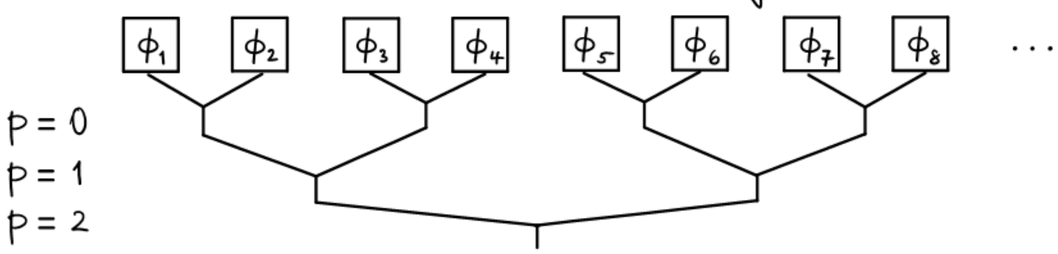
Conjecture ("mass gap" for 2D NLσM): the model is 1-parameter renormalizable (with K as the only relevant coupling) and K continues to decrease under RG flow beyond the perturbative regime.

- Note: dissenting opinion exists (E. Seiler & A. Patrascioiu)
- The MK recursive RG scheme correctly reproduces asymptotic freedom and gives support to the conjecture.

Dyson hierarchical model (exactly renormalizable $\leftarrow$ formulated on a tree).

Field $\phi : \mathbb{Z} \longrightarrow M$ (e.g. $M = \mathbb{Z}_2 = \{\pm 1\}$ for Ising spins)

Hierarchical organization: the energy is a sum of squares (of sums of spins) with interaction constants that are diminished along a tree structure:



III.5 Universality & Scaling

We have seen several examples of renormalization group (RG) flow: the idea is always to pass to a description valid at long distances (or long wavelengths) by integrating out the short-distance physics (modes of short wavelength).

- Language — misnomer: the renormalization "group" has <u>no</u> inverse; it is actually a semigroup. For one example, the Kadanoff block spin map from sites to blocks is not injective. For another, push forward of measures is not reversible in general.

$\longrightarrow$ Think of RG as a "filter".

- Renormalization group flow as a dynamical system:

Imagine a continously varying short-distance cutoff a (for this one may have to switch from real-space to momentum-space renormalization). View renormalization as a dynamical process in the high-dimensional space of all possible couplings:

$$\frac{d}{d\ln a} g_i = \beta_i(g) \qquad i = 1, 2, \dots$$

(very important: autonomous system!)

The β_i are the components of a vector field, but physicists speak simply of the RG beta "function". Note: zeroes of the RG beta function are RG-fixed points.

- Fixed-point "zoology".
  - trivial RG-fixed points: $\begin{cases} \text{high temperature, atomic limit, disordered, } \dots \\ \text{low temperature, mean-field, ordered, } \dots \end{cases}$

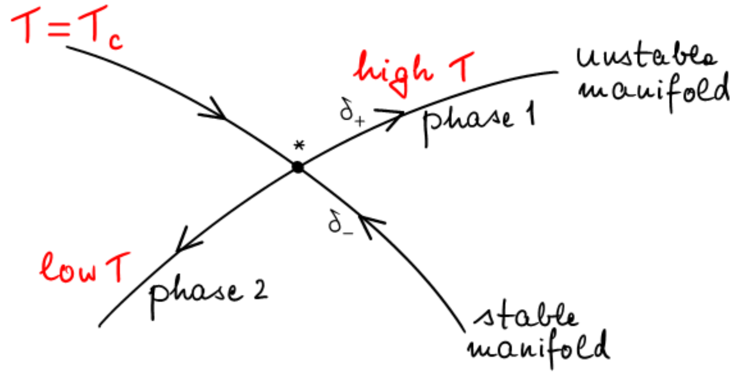
    [alternative methods of treatment exist]
  - non-trivial RG-fixed points: phase transitions ($\to$ continuum field theories).

    [No alternative to RG treatment exists in general.]
  - RG-fixed points (trivial or non-trivial) are "scarce". A given RG-fixed point attracts a large class of different physical systems.

    $\to$ Universality: the basin of attraction of a given RG-fixed point is called a universality class.

- RG flow near fixed point:

The unstable manifold has a small dimension. The stable manifold is also called "critical" manifold.



Scaling law for the correlation length ξ.

Assume one relevant coupling δ_+ (= coordinate generator of unstable manifold) and one irrelevant coupling δ_- (stable manifold).

Linearization of RG flow:
$$\frac{d}{d\ln a}\,\delta_+ = \beta'_+\,\delta_+ + \dots \qquad (\beta'_+ > 0),$$
$$\frac{d}{d\ln a}\,\delta_- = \beta'_-\,\delta_- + \dots \qquad (\beta'_- < 0).$$

Physical observables such as correlation length ξ (CMP) or inverse mass (PP) are RG-invariant: $\xi = \xi\big(\delta_+(a), \delta_-(a); a\big) \underset{a\to a'}{\overset{RG}{=}} \xi\big(\delta_+(a'), \delta_-(a'); a'\big)$.

In the picture of RG as a dynamical system with continously varying cutoff a, RG-invariance implies a differential equation:

$$0 = \frac{d}{d\ln a}\xi = \frac{d\delta_+}{d\ln a}\frac{\partial\xi}{\partial\delta_+} + \frac{d\delta_-}{d\ln a}\frac{\partial\xi}{\partial\delta_-} + \frac{\partial\xi}{\partial\ln a}$$
$$\cong \beta'_+\delta_+\frac{\partial\xi}{\partial\delta_+} + \beta'_-\delta_-\frac{\partial\xi}{\partial\delta_-} + \xi.$$

since ξ is a length

Close to the critical manifold and for large enough cutoff a we may neglect the irrelevant coupling (put $\delta_- = 0$). Then

$$d\ln\xi = \frac{d\xi}{\xi} = -\frac{1}{\beta'_+}\frac{d\delta_+}{\delta_+} = d\ln\delta_+^{-1/\beta'_+} \quad \curvearrowright$$

$$\xi = \text{const}\cdot\delta_+^{-1/\beta'_+} \sim |T-T_c|^{-\nu} \quad \text{where } \nu = 1/\beta'_+ \text{ and } \delta_+ \sim T-T_c.$$

Lecture 19.

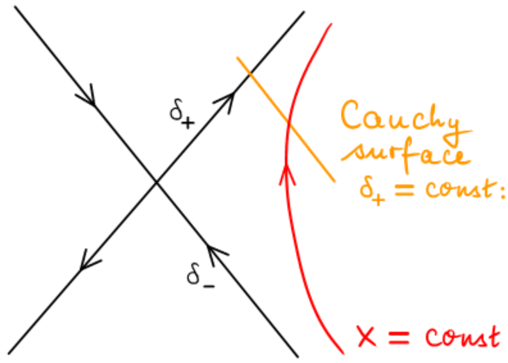
- Scaling (cont'd). Recall $0 = \frac{d}{d\ln a}\xi = \beta'_+\delta_+\frac{\partial\xi}{\partial\delta_+} + \beta'_-\delta_-\frac{\partial\xi}{\partial\delta_-} + \xi$.

  Neglecting the irrelevant coupling δ_- one obtains $\xi = \text{const}\cdot a\,\delta_+^{-1/\beta'_+}$.

Now take δ_- into account and compute the leading correction to the scaling law for the correlation length (solve the PDE by the method of characteristics):

$$\frac{d}{d\ln a}\delta_\pm = \beta'_\pm\delta_\pm \rightsquigarrow \delta_\pm \propto a^{\beta'_\pm}.$$

Note: $x \equiv \delta_+^{1/\beta'_+}\,\delta_-^{-1/\beta'_-}$ is constant along the trajectories of the RG flow.



$$F(x) = \xi / a\,\delta_+^{-1/\beta'_+}$$

Solution parametrized by scaling function $F(x)$:

$$\xi = a\,\delta_+^{-1/\beta'_+}F(x) = a\,\delta_+^{-1/\beta'_+}F\left(\delta_+^{1/\beta'_+}\,\delta_-^{1/|\beta'_-|}\right).$$

If $F(x)$ is analytic in x at $x=0$, and $F(x) = F_0 + F_1 x + \mathcal{O}(x^2)$,

then $\xi/a = F_0\,\delta_+^{-1/\beta'_+} + F_1\,\delta_-^{1/|\beta'_-|} + \dots$

leading corrections to scaling

III.6 Migdal–Kadanoff renormalization of 2D Heisenberg spin model

Consider classical spins ($n \in \mathbb{R}^3$, $n^2 = 1$) with ferromagnetic coupling on a square lattice (cf. Exercise Sheet 09).

Statistical weight associated with pair n, n' of neighbor spins $= e^{K n\cdot n'} \equiv \Omega_{\text{initial}}(n\cdot n')$.

Migdal–Kadanoff recusion step:

$$\Omega_{\text{old}}(n\cdot n') \xrightarrow[\text{moving}]{\text{bond}} \Omega_2(n\cdot n') = \Omega_{\text{old}}(n\cdot n')^2$$

$$\xrightarrow[\text{mation}]{\text{deci-}} \int d^2 n'' \, \Omega_2(n\cdot n'')\, \Omega_2(n''\cdot n') \equiv \Omega_{\text{new}}(n\cdot n').$$

Here, implement the recursive scheme analytically in the low-temperature regime (large K).

Idea: expand around Gaussian fixed point.

Warm-up. $x \in \mathbb{R}$:

$$e^{-(x-x')^2/4t} \xrightarrow[\text{moving}]{\text{bond}} e^{-(x-x')^2/2t} \xrightarrow[\text{mation}]{\text{deci-}} \sqrt{\pi t}\; e^{-(x-x')^2/4t} \quad \text{(fixed point for any } t\text{)}.$$

Write $n\cdot n' \equiv \cos\theta$ (θ polar angle of n' with respect to n; also: geodesic distance on S^2).

$$e^{K n\cdot n'} = e^{K\cos\theta} \doteq \text{const}\cdot e^{-K\theta^2/2} \propto e^{-\theta^2/4t} \quad (t^{-1} = 2K).$$

Q: How to do the convolution integral for the decimation step?

A: The heat kernel, p_t, defined as the solution of $\partial_t p_t = \Delta^{\text{rad}}_{S^2} p_t$ (diffusion eqn) with initial condition $\lim_{t\to 0+} p_t(\theta) = \delta(\theta)$, has the semigroup property $p_t * p_t = p_{2t}$ under convolution ($*$).

Short-time expansion of heat kernel: $p_t(\theta) = \dfrac{e^{-\theta^2/4t}}{4\pi t}\left(1 + \dfrac{\theta^2}{12} + \ldots\right)$.

Now, $e^{-\theta^2/4t} \propto p_{t - t^2/3}(\theta)$ $\curvearrowright$ transform $t \longmapsto t - t^2/3$.

Hence the effect on $e^{-\theta^2/4t}$ of convolution (up to leading nonlinear order in t)

is $\quad t \xmapsto{\text{transform}} t - t^2/3 \xmapsto{\text{convolution}} 2(t - t^2/3) \xmapsto[\text{transform}]{\text{inverse}} 2t + 2t^2/3$.

$\curvearrowright$ Migdal–Kadanoff RG (cutoff $a \longrightarrow 2a$):

$$t \xmapsto[\text{moving}]{\text{bond}} t/2 \xmapsto[\text{mation}]{\text{deci-}} t + t^2/6 + \mathcal{O}(t^3).$$

Conversion to the continuous picture (equivalent for small t):

$$\frac{d}{d\ln a} t = b t^2 + \mathcal{O}(t^3), \quad b = 1/6\ln 2 > 0 \quad \text{"asymptotic freedom".}$$

An important physical consequence is ($\rightarrow$ mass gap conjecture) "dynamical mass generation":

correlation length $\xi(t;a)$ (spin-spin correlation function) satisfies

$$0 = \frac{d}{d\ln a}\xi = \frac{dt}{d\ln a}\frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial\ln a} = b t^2 \frac{\partial\xi}{\partial t} + \xi$$

$$\curvearrowright \quad d\ln\xi = \frac{d\xi}{\xi} = -\frac{1}{b}\frac{dt}{t^2} = d\,\frac{1}{bt}.$$

Integrate & exponentiate $\curvearrowright$ $\xi = \text{const}\cdot\exp(1/bt)$. Note: the result for ξ is non-analytic (!) in the small parameter t of our perturbative calculation.

Comment: mass $m \sim \xi^{-1} \sim e^{-1/bt}$ is nonzero even though the classical field theory $S = \frac{1}{t}\int d^2x\,(\nabla n)^2$ is <u>massless</u>!

Info: $\exists$ strong analogy with 4D non-Abelian gauge theory (Yang-Mills).

Generalization to Riemannian target spaces other than S^2:

The metric tensor expands in Riemann normal coordinates as

$$g_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{1}{3}R_{\mu\sigma\nu\tau}\,x^\sigma x^\tau + \mathcal{O}(|x|^3).$$

Riemann curvature tensor

Can use this to derive the short-time expansion of the heat kernel ...

Lecture 20

Addendum: dynamical mass generation

$$S = \frac{1}{t} \int d^2x \, (\nabla n)^2$$

is a massless theory as a classical field theory (spin waves as Goldstone bosons).

The effect of quantum (or statistical) fluctuations in 2D is to make the spin-spin correlation function decay exponentially, $\langle n(r)\, n(r') \rangle \sim e^{-|r-r'|/\xi}$, thereby generating a mass $m \sim \xi^{-1}$ not present in the classical theory.

Info 1. is consistent with Mermin-Wagner-Coleman theorem: no spontaneous breaking of continuous symmetries (of compact type) in two dimensions.
(Note: MWC permits algebraic decay of correlations; is weaker statement than mass gap conjecture)

Info 2.

Antiferromagnetic quantum spin chain (1D)

$$H = J \sum_n \mathbf{S}_n \cdot \mathbf{S}_{n+1} \qquad (J>0)$$

for Spin ½ is known to be massless (by the Lieb–Schultz–Mattis Thm, 1964).

Haldane considered (1983/84) the case of spin $|S| = 1$ and argued that the model is massive (exponential decay of correlations)!

His argument was to show that the low-energy physics of the $|S|=1$ chain is given by the O(3) nonlinear σ model with $t^{-1} \sim |S|$.

⇝ Nobel Prize for Physics 2016
(shared with Kosterlitz & Thouless)

III.7 Kosterlitz–Thouless transition

[one more real-space RG ...]

KT 1973; K (RG) 1974
(mean field)

Planar model aka xy-model in 2D:

$(J>0)$

$$Z = \int e^{-\beta H}, \quad H = -J \sum_{\langle nn' \rangle} \cos(\theta_n - \theta_{n'})$$

(nearest neighbors on square lattice)

θ = phase of superfluid order parameter
("London approximation")

Looks easy, but is not so easy to analyze!

<u>Picture to be established:</u>

① At low temperature:

spin waves (perturbed by bound vortex-antivortex pairs)

$\exists$ line of RG-fixed points with algebraic decay of correlation function $\langle e^{i\theta_n} e^{-i\theta_{n'}} \rangle$

② At high temperature:

gas of unbound vortices and antivortices;

$\sim$ exponential decay of correlation function.

(3) Estimate of transition temperature

At what temperature is the energy cost of creating a vortex balanced by the gain in entropy?

Single vortex centered at (x_0, y_0) (in continuum notation)

$$d\theta = \frac{(x-x_0)\,dy - (y-y_0)\,dx}{(x-x_0)^2 + (y-y_0)^2} \qquad \underline{\underline{dy = * dx}} \quad dx = - * dy$$

$$= * \frac{dr_0}{r_0} \qquad \left(r_0 = \sqrt{(x-x_0)^2 + (y-y_0)^2} \right)$$

$$= d\phi_0 \qquad \text{(angular form w.r.t. } (x_0, y_0))$$

$$\int d^2x \, (\nabla\theta)^2 = \int \frac{dr_0}{r_0} \wedge d\phi_0 = 2\pi \ln(L/a)$$

L = system size, a = lattice constant.

Energy cost: $e^{-\beta H} \propto e^{-\frac{1}{2}\beta J \int d^2x\,(\nabla\theta)^2} = e^{-\pi\beta J \cdot \ln(L/a)} = (L/a)^{-\pi\beta J}$.

Entropy gain: $\sum_{(x_0,y_0)} 1 = (L/a)^2$

critical temperature: $2 = \pi \beta_c J$, $\beta_c = \frac{1}{k_B T_c}$.

Short note on topology.

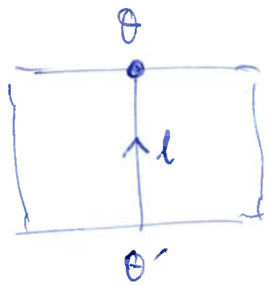
In the continuum picture a vortex (antivortex) at position p is a field configuration $\theta : \mathbb{R}\setminus\{p\} \longrightarrow \mathbb{R}/2\pi\mathbb{Z} \cong S^1$ of winding number one (minus one). (cf. Heisenberg spins)

Its energy $\int d^2x \, (\nabla\theta)^2$ would be infinite (due to the fast variation of θ near p) if there was no short-distance cutoff. (Anti-) vortices carry the potential to "disorder" the spin system (in fact, that's what they do a high temperature, $T > T_c$).

Separating vortices from spin waves

$$e^{\beta J \cos(\theta-\theta')} \underset{\text{approximation}}{\overset{\text{Villain}}{\cong}} \text{heat kernel on } S^1$$

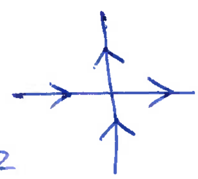
$$= \sum_{l\in\mathbb{Z}} e^{il(\theta-\theta')} \, e^{-l^2/2\beta J} \cdot \text{const}$$



Partial "integration": $\sum_{\text{links}} l \cdot \text{grad}\,\theta = - \sum_{\text{sites}} \theta \, \text{div}\, l$.

$$\int d\theta \; e^{-i \sum_{\text{sites}} \theta \, \text{div}\, l} = 0 \quad \text{unless div}\, l \equiv 0$$

cf. Kirchhoff's first rule



Now $Z = \int e^{-\beta H} \cong \text{const} \cdot \sum_{\text{div}\, l = 0} e^{-\frac{1}{2\beta J} \sum_{\text{links}} l^2}$

$\operatorname{div} l = 0 \rightsquigarrow l = \operatorname{rot} m$??

[Self-inflicted adversity!

Lattice exterior calculus (chains & cochains) would be appropriate here, in particular in view of generalizations to other lattices, higher dimensions and higher-rank fields.]

Info.

θ 0-cochain

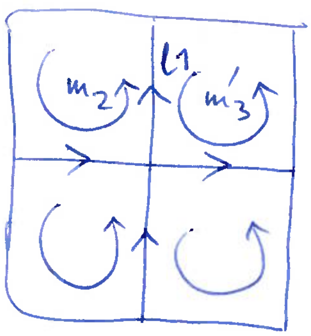
$d\theta$ 1-cochain ($\operatorname{grad}\theta$)

l 1-chain

m 2-chain : $\partial m = l \rightsquigarrow \partial l = 0$.

$$\langle \underbrace{l, d\theta}_{\operatorname{grad}\theta} \rangle \equiv \langle \underbrace{\partial l}_{-\operatorname{div} l}, \theta \rangle$$ (∂ boundary operator)

Pictorially:



"loop currents" m

$l = \partial m$ means

$l_1 = m_2 - m_3$ etc.

$$Z = \text{const.} \sum_{2\text{-chains } m} e^{-\frac{1}{2\beta J} \underbrace{\sum_{\text{links}} (\partial m)^2}}$$

$$= \sum_{\text{plaqs}} m(-\Delta m)$$

Laplace on 2-chains

Expression poor for low temperatures (large β).

$\rightsquigarrow$ Use Poisson summation formula.

Lecture 21 (KT transition cont'd)

Recall duality transformation (note $\beta J \to 1/\beta J$)

$$Z = \int d\theta \; e^{\beta J \sum_{\text{links}} \cos(\theta_n - \theta_{n'})}$$

$$\overset{\text{Villain}}{\cong} \quad \text{const.} \sum_{\{m\}} e^{-\frac{1}{2\beta J} \sum_{\text{plaqs}} m_p (-\Delta m)_p} .$$

For low T (high β) use Poisson summation formula (warning: $m \neq m$)

$$\sum_{m \in \mathbb{Z}} f(m) = \int_{\mathbb{R}} d\phi \, f(\phi) \sum_{m \in \mathbb{Z}} e^{2\pi i m \phi} .$$

Then $\boxtimes$

$$Z \propto \int d\phi \sum_{\{m\}} e^{-\frac{1}{2\beta J} \sum_{\text{plaqs}} \phi_p (-\Delta \phi)_p + 2\pi i \underbrace{\langle m, \phi \rangle}_{\text{pairing } \langle \text{2-chain, 2-cochain} \rangle}}$$

<u>Info.</u> Is 2D variant of what is known as "boson-vortex duality" in 2+1 dimensions.

<u>Remark.</u> Can integrate out ϕ $\leadsto$

$$Z \propto \sum_{\{m\}} e^{-2\pi^2 \beta J \sum_{\text{plaqs}} m_p (-\Delta^{-1} m)_p} .$$

$\dim \ker \Delta = 1$ ($\ker \Delta$ = constant 2-chains) $\leadsto$ $\sum m_p = 0$ ("charge neutrality")

Note $(-\Delta)^{-1}(r, r') \overset{2D}{\sim} \ln |r - r'|$.

<u>Interpretation</u>: neutral plasma of charged vortices (m_p) with long-range (log) interactions.

Screening? Yes, non-trivial. Needs RG!

Advanced perspective (from continuum picture):

On $\mathbb{R}^2 \setminus \{p_1, \dots, p_N\}$ (vortex singularities removed)

make Hodge decomposition:

$$d\theta = \text{harmonic} + \text{exact} + (\text{co-exact})$$

$$= \sum_{j=1}^{N} m_j \tau_{p_j} + d\phi \qquad (\tau_p \text{ angular 1-form w.r.t. } p)$$

$$= \text{vortex part} + \text{spin wave part}$$

(these decouple in Villain approximation)

Setting up the RG. Start from $\boxtimes$.

Info: RG generates vortex chemical potential
(→ fugacity)

Therefore, add term $\lambda \sum_p m_p^2$ to initial energy function.

Anticipate that only $m_p = 0, \pm 1$ make significant contribution (λ large!):

$$\sum_{m=0,\pm 1} e^{-\lambda m^2 + 2\pi i m \phi} = 1 + e^{-\lambda} 2\cos 2\pi\phi \simeq e^{2e^{-\lambda}\cos 2\pi\phi} .$$

Scale $\phi \to \sqrt{\beta J}\,\phi$.

Effective action (continuum approximation)

$$S = \frac{1}{2}\int d^2r\, \phi(-\Delta\phi) - \mu \int d^2r\, \cos(2\pi\sqrt{\beta J}\,\phi)$$

$$\mu = 2e^{-\lambda}/a^2 .$$

(starting point for renormalization)
compute RG flow of β, μ.

Implement RG by momentum-shell integration.

initial cutoff Λ, reduced cutoff $\Lambda' < \Lambda$.

$$\phi(r) = \int \frac{d^2k}{(2\pi)^2} \tilde{\phi}(k)\, e^{ikr} = \underbrace{\varphi}_{\text{from } 0<|k|<\Lambda'} + \underbrace{h}_{\text{from } \Lambda'<|k|<\Lambda}.$$

Decomposition is L^2-orthogonal:

$$\int d^2r\, \phi(-\Delta\phi) = \int d^2r\, \varphi(-\Delta\varphi) + \int d^2r\, h(-\Delta h).$$

Do the Gaussian integral over h

$$\left\langle e^{\mu \int d^2r \cos\left(2\pi\sqrt{\beta}(\varphi+h)\right)} \right\rangle_h \qquad (\text{simplified } \beta\partial \equiv \beta)$$

$$\overset{\mu \text{ small}}{=} 1 + \mu \int d^2r \cos\left(2\pi\sqrt{\beta}\,\varphi(r)\right) \underbrace{\left\langle e^{\pm 2\pi i \sqrt{\beta}\, h(r)} \right\rangle_h}_{= A(0) \text{ where } A(r) = e^{-2\pi^2\beta G_h(r)},}$$

$\to$ renormalization of μ

$$G_h(r-r') = \int_{\Lambda'<|k|<\Lambda} \frac{d^2k}{(2\pi)^2} \frac{e^{ik(r-r')}}{k^2}.$$

[Follow JB Kogut, Rev Mod Phys. 51 (1979)]

Note cumulant expansion:

$$\ln\langle e^X\rangle = \ln\left(1 + \langle X\rangle + \tfrac{1}{2}\langle X^2\rangle + \dots\right)$$

$$= \langle X\rangle + \underbrace{\tfrac{1}{2}\langle X^2\rangle - \tfrac{1}{2}\langle X\rangle^2}_{= \frac{1}{2}\langle X^2\rangle_{\text{cum.}}} + \dots$$

Exercise. $\left\langle \left(\int d^2r \, \cos(2\pi\sqrt{\beta}(\varphi+h)) \right)^2 \right\rangle_{conn}$

$$= \frac{1}{2} \int d^2r \int d^2r' \left\{ \cos\left[2\pi\sqrt{\beta}(\varphi(r)+\varphi(r'))\right] A^2(0)\left(A^2(r-r')-1\right) \right.$$
$$\left. + \cos\left[2\pi\sqrt{\beta}(\varphi(r)-\varphi(r'))\right] A^2(0)\left(A^{-2}(r-r')-1\right) \right\}$$

If $A^{\pm 2}(r-r')-1$ appreciable only $|r-r'|$ small (since $G_h(r-r')$ falls off), then can expand:

$$\varphi(r)-\varphi(r') \cong (r-r')\cdot \operatorname{grad}\varphi\left(\tfrac{r+r'}{2}\right).$$

So $\{\ldots\} \cong A^2(0)\left(A^2(r-r')-1\right)\cos\left(4\pi\sqrt{\beta}\,\varphi\left(\tfrac{r+r'}{2}\right)\right)$
$$+ A^2(0)\left(A^{-2}(r-r')-1\right)\left(1-2\pi^2\beta\left[(r-r')\cdot\operatorname{grad}\varphi\right]^2+\ldots\right).$$

Neglect first term ($\propto \mu^2$, negligible w.r.t. $\mu \to \mu A(0)$ of above) but keep second term, which [after integration over $(r-r')$ and re-exponentiation according cumulant expansion] gives a correction to $\frac{1}{2}\int d^2r\, \varphi(-\Delta\varphi)$. Rescaling φ to preserve $\frac{1}{2}$ one gets a renormalization of β.
By specializing to an infinitesimal change $\Lambda \to \Lambda' = \Lambda - d\Lambda$ one arrives at RG flow equations for β, μ (cf. Lect. 22).

- CAVEAT. Things are not as easy as it seems!
  Sharp momentum cutoff $\Lambda' < |k| < \Lambda$ gives SLOW fall off of G_h in real space. A more involved scheme using a <u>smooth</u> momentum cutoff is called for.

Lecture 22

RG-treatment of Kosterlitz–Thouless transition (conclusion):

Recall $S = \frac{1}{2}\int d^2r\, \phi(-\Delta\phi) - \mu \int d^2r \cos(2\pi\sqrt{\beta}\,\phi)$.

RG-strategy: integrate out modes in momentum shell $\Lambda' < |k| < \Lambda$ by means of a cumulant expansion in $\phi - \varphi = h$.

– First cumulant ($\to$ cosine term)

$$\mu' = \mu\, A(0), \qquad A(r-r') = e^{-2\pi^2\beta\, G_h(r-r')},$$

$$G_h(\xi) = \int_{\Lambda'<|k|<\Lambda} \frac{d^2k}{(2\pi)^2}\, \frac{e^{ik\xi}}{k^2}.$$

– Second cumulant. Keep

$$\frac{\mu^2}{2}\,\frac{1}{2}\int d^2r \int d^2r'\, A^2(0)\left(A^{-2}(r-r')-1\right)\left[(r-r')\cdot \operatorname{grad}\varphi\right]^2(-2\pi^2\beta)$$

$$= -\frac{1}{2}\mu^2\beta\pi^2 \int \varphi(-\Delta\varphi)\cdot A^2(0) \underbrace{\int d^2\xi\, \left(A^{-2}(\xi)-1\right)\xi^2/2}_{\equiv\, a_2 \ (<\infty \text{ for smooth cutoff})}$$

Changes the stiffness $\frac{1}{2}$ of the Gaussian free field

$$\frac{1}{2} \longrightarrow \frac{1}{2} + \frac{1}{4}\pi^2\beta\mu^2 A^2(0)\, a_2 \equiv \frac{1}{2} Z.$$

Scale $\varphi \longrightarrow \varphi/\sqrt{Z}$. Then

$$\beta' = \beta/Z = \beta/\left(1+\frac{1}{2}\pi^2\beta\mu^2 A^2(0)\, a_2\right) \blacksquare$$

Specialize to infinitesimal momentum shell ($\Lambda' = \Lambda - d\Lambda$) to obtain differential eqn for RG flow.

① $$\frac{d\mu}{d\Lambda} = \mu(-2\pi^2\beta)\underbrace{\frac{d}{d\Lambda}G_\mu(0)}_{=\frac{1}{(2\pi)^2}\cdot\frac{2\pi\Lambda}{\Lambda^2}} = -\mu\pi\beta/\Lambda .$$

② Similar for $\frac{d\beta}{d\Lambda}$ ($\to$ Exercise).

Standardization. Introduce

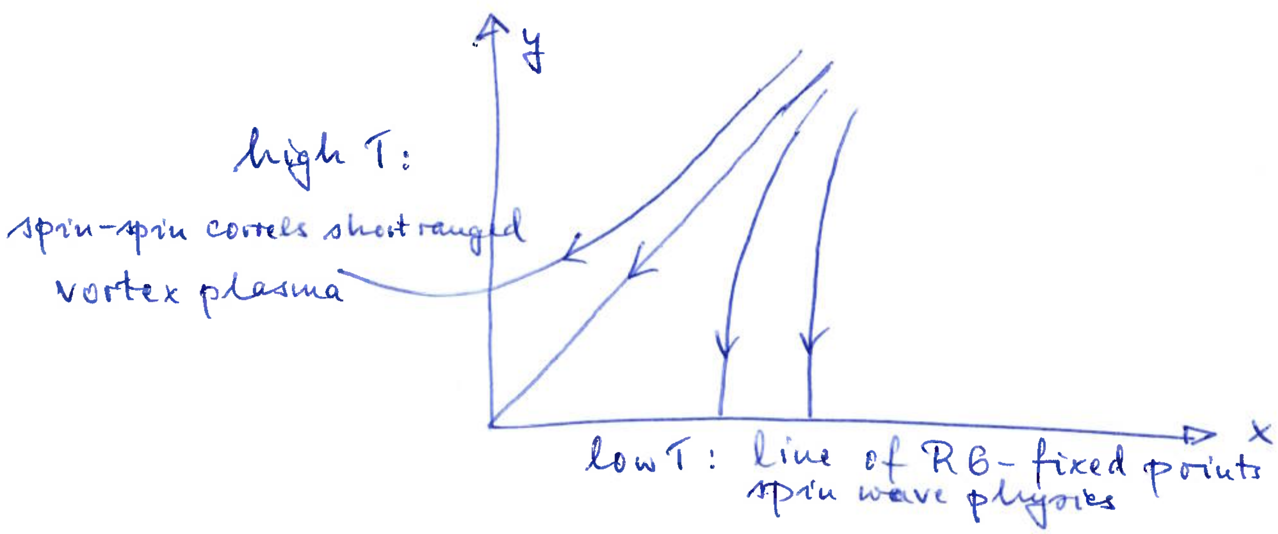
$$y = \mu a^2, \quad x = \pi\beta - 2 \quad (\text{so } x_{crit} = 0),$$

$a = 1/\Lambda$ and adjust the scale of a.

Then "standard" form of differential RG flow,

$$\frac{d\,x^2}{d\ln a} = -2xy^2 = \frac{d\,y^2}{d\ln a}.$$

Note: RG-invariant $x^2 - y^2 \rightsquigarrow$ hyperbolic flow in xy-plane



Peculiarity: correlation length $\xi(T) \sim \exp\left(\frac{b}{(T-T_c)^{1/2}}\right)$ has reason $\frac{d\tau^2}{d\ln a} \sim \tau^3$ (imbalance of powers).

[Anecdote]

Lecture 22 — Part C

III.8 Vertex functions & effective action

Recall from Chapter I (Perturbation Theory),
Section I.5 for

scalar field ϕ with action functional $S[\phi]$:

Generating functional for connected Green's functions

$$F[j] = \ln Z[j] = \ln \int \mathcal{D}\phi \; e^{-S[\phi] + \int d^d x \, j(x) \phi(x)} .$$

Legendre transform to generating functional for n-point vertex functions:

$$\Gamma[\varphi] = \int d^d x \, j(x) \varphi(x) - F[j] \quad \text{where} \quad j = j[\varphi]$$

IDEA: we may take j, φ to be slowly varying
(not containing high-momentum modes)

CLAIM. Replacing $S[\phi]$ by $\Gamma[\varphi]$ and computing at tree level gives the full quantum theory.

($\rightsquigarrow$ Interpretation of $\Gamma[\varphi]$ as effective action
for the long wavelength physics)

Proof. Treating $e^{-\Gamma[\varphi] + \int j\varphi}$ at tree level,

i.e. as a classical theory (with no quantum fluctuations for the field φ) gives the equations of motion

$$\frac{\delta}{\delta \varphi(x)} \Gamma[\varphi] = j(x) .$$

These invert the Legendre transform $F \to \Gamma$,

$\Gamma = \int j\varphi - F$ from $\frac{\delta}{\delta j(x)} F[j] = \varphi(x)$.

Hence $e^{-\Gamma + \int j \varphi} = e^{F} = e^{\ln Z}$

$$= Z[j] = \int \mathcal{D}\phi \, e^{-S[\phi] + \int j\phi}$$

Comment: provides foundation for Landau theory.

III. 9 One-loop effective action from background field method.

Recall $Z[j] = \int \mathcal{D}\phi \; e^{-S[\phi] + \int j\phi} = e^{F[j]}$

and $F[j] = \int j\varphi - \Gamma[\varphi]$ (Legendre transform $F \to \Gamma$).

Write this as

$$e^{-\Gamma[\varphi]} = e^{F[j] - \int j\varphi} = \int \mathcal{D}\phi \; e^{-S[\phi] + \int j(\phi - \varphi)} .$$

Language: φ "background field"

$h = \phi - \varphi$ "fast field" (← quantum fluctuations)

Change integration variables from ϕ to $h = \phi - \varphi$ and use $j = \frac{\delta\Gamma}{\delta\varphi}$. Then

$$e^{-\Gamma[\varphi]} = \int \mathcal{D}h \; e^{-S[\varphi + h] + \int \frac{\delta\Gamma}{\delta\varphi} h} .$$

Now expand:

$$S[\varphi + h] = S[\varphi] + \frac{\delta S}{\delta\varphi} \cdot h + \frac{1}{2} h \cdot \frac{\delta^2 S}{\delta\varphi^2} \cdot h + \mathcal{O}(h^3).$$

Also, let $\Gamma = S + K$ (thus, K is the correction that turns to bare action $S[\varphi]$ into the effective action $\Gamma[\varphi]$). Set up equation for K:

$$e^{-K[\varphi]} = e^{S[\varphi]} e^{-\Gamma[\varphi]} = e^{S[\varphi]} \int \mathcal{D}h \; e^{-S[\varphi+h] + \int \frac{\delta(S+K)}{\delta\varphi} h}$$

$$= \int \mathcal{D}h \; e^{-\frac{1}{2} h \cdot \frac{\delta^2 S}{\delta\varphi^2} \cdot h + \mathcal{O}(h^3) + \frac{\delta K}{\delta\varphi} \cdot h}$$

In the so-called one-loop approximation, one neglects K on the right-hand side. Then

One – loop approximation:

$$K[\varphi] = -\ln \int \mathcal{D}h \; e^{-\frac{1}{2} h \cdot \frac{\delta^2 S}{\delta \varphi^2} \cdot h}$$

$$= + \ln \mathrm{Det} \left(\frac{\delta^2 S}{\delta \varphi^2} \right) = \mathrm{Tr} \ln \left(\frac{\delta^2 S}{\delta \varphi^2} \right).$$

Example. $S = \int d^d x \left(\frac{\mu}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right).$

$$\frac{1}{2} h \cdot \frac{\delta^2 S}{\delta \varphi^2} \cdot h = \frac{1}{2} \int d^d x \left(\mu (\nabla h)^2 + m^2 h^2 + \frac{1}{2} \varphi^2 h^2 \right).$$

$$\rightsquigarrow K[\varphi] = \frac{1}{2} \ln \mathrm{Det} \left(-\mu \Delta + m^2 + \varphi^2/2 \right).$$

Remark. Easy to calculate for $\varphi = \varphi_0 = \text{const}$ or in one dimension (see QFT-1).
More on this subject in L25.

Info. Similar treatment possible for fermions (say, complex).

$$Z[\zeta, \bar{\zeta}] = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \; e^{-S[\bar{\Psi}, \Psi] + \int (\bar{\zeta} \Psi + \bar{\Psi} \zeta)}$$

anticommuting source fields $\zeta, \bar{\zeta}$.

$F = \ln Z$, as before.

$$\frac{\delta}{\delta \bar{\zeta}(x)} F \equiv \psi(x), \quad \frac{\delta}{\delta \zeta(x)} F \equiv -\bar{\psi}(x).$$

$$\Gamma[\psi, \bar{\psi}] = \int (\bar{\zeta} \psi + \bar{\psi} \zeta) - F[\zeta, \bar{\zeta}].$$

CHECK

$$\frac{\delta \Gamma}{\delta \bar{\psi}(x)} = \zeta(x) - \frac{\delta \zeta}{\delta \bar{\psi}(x)} \cdot \bar{\psi} - \frac{\delta \zeta}{\delta \bar{\psi}(x)} \cdot \frac{\delta F}{\delta \zeta} + \frac{\delta \bar{\zeta}}{\delta \bar{\psi}(x)} \cdot \psi - \frac{\delta \bar{\zeta}}{\delta \bar{\psi}(x)} \cdot \frac{\delta F}{\delta \bar{\zeta}}$$

$$= \zeta(x) + 0 \checkmark$$

III. 10 Landau–Ginzburg (–Wilson) Theory (a.k.a. mean field)

Assume the existence of an order parameter field;
e.g. magnetization m.

$m = 0$ for $T > T_c$ and $m \neq 0$ for $T < T_c$
(spontaneous symmetry breaking occurs)

Consider the "low-energy effective" action $\Gamma[m]$ (may be hard if not impossible to calculate but should still exist).

For T near T_c can expand Γ in the small quantity m.

Write down all terms allowed by symmetry (and other considerations if applicable)

Example. $\Gamma[m] = \int d^dx \left(\frac{t}{2} m^2 + u\, m^4 + \frac{K}{2} (\nabla m)^2 - h \cdot m + \dots \right)$

Find the thermodynamic state by minimization of the free energy $\Gamma[m = m_0]$, $m_0 = \text{const}$.

For $h = 0$: $m \sim \sqrt{-t} \sim \sqrt{T_c - T}$ from $0 = \left.\frac{\partial \Gamma}{\partial m_0}\right|_{h=0}$

mean-field critical behavior of magnetization

Correlation length $\xi \sim |T - T_c|^{-1/2}$ ($\nu = \frac{1}{2}$, independent of dimension).

For $T > T_c$: $\Gamma = \frac{1}{2} \int d^dx \left(K (\nabla m)^2 + \underbrace{(T - T_c)}_{\sim \xi^{-2}} m^2 \right)$

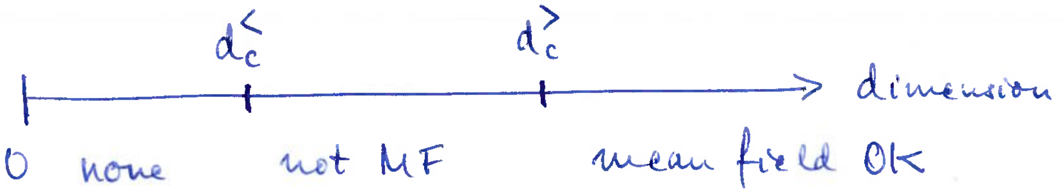
Ginzburg criterion (qualitative).

Mean-field approximation valid if fluctuations

$$\frac{\langle m^2\rangle - \langle m\rangle^2}{\langle m\rangle^2} \quad \text{are small}$$

$\rightsquigarrow$ G. criterion, which determines upper critical dimension.

Picture: critical behavior



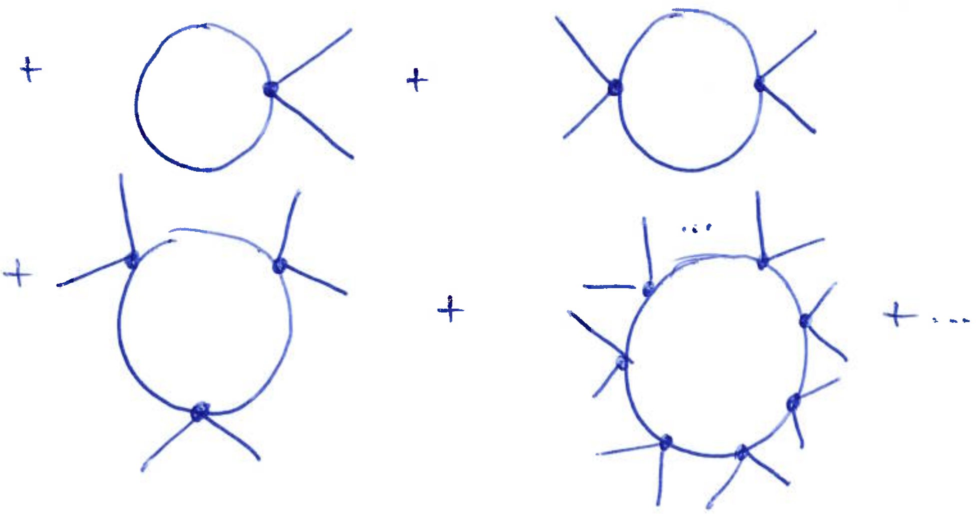
Ginzburg criterion (quantitative).

Recall the one-loop approximation to the effective action of ϕ^4-theory $S[\phi] = \int d^dx \left(\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right)$ $(\mu=1)$:

$$\Gamma[\varphi] = S[\varphi] + K[\varphi], \quad K[\varphi] = +\frac{1}{2}\operatorname{Tr}\ln\left(-\Delta + m^2 + \frac{\lambda}{2}\varphi^2\right).$$

Remark. K expands in the coupling λ as

$K[\varphi] = \text{const} +$



Thus, an alternative way of obtaining $K[\varphi]$ is to sum all one-loop graphs ▪

Susceptibility (for $T \searrow T_c$) χ:

$$\chi^{-1} = m^2 + \frac{\lambda}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}; \qquad m^2 = T - T_c^{MF}.$$

Note that the mean-field critical temperature gets shifted to a lower value given by

$$0 \equiv \chi^{-1} = m_c^2 + \frac{\lambda}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2} \qquad \text{(can neglect } m^2 \text{ in the denominator).}$$

So $\chi^{-1} = (m^2 - m_c^2) + \frac{\lambda}{2}\int \frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2+m^2} - \frac{1}{k^2}\right).$

Now, $m^2 - m_c^2 = T - T_c$ and

$$\frac{1}{m^2+k^2} - \frac{1}{k^2} = \frac{-m^2}{k^2(k^2+m^2)},$$

hence $$\chi^{-1} = T - T_c - \frac{\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\,\frac{m^2}{k^2(k^2+m^2)}.$$

In Landau theory one has $\chi^{-1} = m^2 = T - T_c^{MF}$.

Criterion for (in-) validity (Ginzburg):

$$\frac{\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2(k^2+m^2)} \approx 1.$$

The k-integral is finite (in the infrared, k small) for $d>4$. $\curvearrowright$ Upper critical dimension $d_c^> = 4$ ☐

L24: III. 11 Background-field RG for nonlinear σ models

Info (not in some modern textbooks).

By "nonlinear model" one means a (in 2D) field theory of maps

$$\phi : \Sigma = \mathbb{R}^2 \text{ (Euclidean plane)} \longrightarrow M \text{ (Riemannian manifold)},$$

M with metric tensor $= g_{ij}\, du^i du^j$ (local coordinate fcts $u^i : M \to \mathbb{R}$).

Action functional $S[\phi] = \frac{1}{T} \int d^2r \; g_{ij}(\phi(r))\, \nabla\phi^i(r) \cdot \nabla\phi^j(r)$,

$\phi^i(r) \equiv u^i(\phi(r))$;

$$Z = \int \mathcal{D}\phi \; e^{-S[\phi]}.$$

Strong result due to D.H. Friedan (Ann. Phys. 1985):

$$\frac{d}{d\ln a}\left(\frac{g_{ij}}{T}\right) = \underbrace{-\,\mathrm{Ric}_{ij}}_{\text{one-loop}} + \underbrace{-\,\frac{T}{2} R_{ipqr} R_j{}^{pqr}}_{\text{two-loop}} + \mathcal{O}(T^2) + \ldots$$

where R_{ijkl} = Riemann curvature tensor of (M, g)

and $\mathrm{Ric}_{ij} = R^k{}_{ikj}$ Ricci curvature tensor.

Comment: ① Friedan's result simplifies when M is a symmetric space (in which case one speaks of a nonlinear σ model), where $g_{ij} \propto \mathrm{Ric}_{ij} \propto R_{ipqr} R_j{}^{pqr}$.

② For M a symmetric space of compact type (e.g. $M = S^2$ with the round geometry) one has $\mathrm{Ric}_{ij} = b\, g_{ij}$, $b > 0$.

In that case, $\dfrac{d}{d\ln a}\left(\dfrac{1}{T}\right) = -b + \mathcal{O}(T)$, or equivalently,

$$\frac{d}{d\ln a} T = +bT^2 + O(T^3) \qquad (\text{"asymptotic freedom"})$$

(cf. Sect. III.6, Migdal–Kadanoff RG scheme).

Goal (of the present section):

Verify the RG beta function in one-loop approximation for the case of $M = S^2$ (actually, any symmetric space M) by utilizing the background field method.

Recall from Sect. III.9 (background field method for linear models) the decomposition

$$\phi = \varphi + h$$

(φ background, "slow"; h quantum, "fast").

Q: how to make a similar decomposition in the nonlinear setting (with target space M), where "+" is not available?

[A (secret): do math on the tangent bundle TM].

Specifically, for $M = S^2$ we can proceed as follows.

$$n \in S^2 \sim \sum_{i=1}^{3} n_i \sigma_i = \begin{pmatrix} n_3 & n_1 - i n_2 \\ n_1 + i n_2 & -n_3 \end{pmatrix} \equiv Q,$$

$$Q = Q^\dagger, \quad \mathrm{Tr}\, Q = 0, \quad Q^2 = (n_1^2 + n_2^2 + n_3^2)\cdot 1 = 1.$$

Write $Q = g\,\sigma_3\, g^{-1}$, $g \in SU(2)$.

Since $k\,\sigma_3\, k^{-1} = \sigma_3$ for $k = e^{i\theta\sigma_3} \in U(1) \equiv K$,

this parametrization realizes the 2-sphere as a coset space $S^2 = SU(2)/U(1)$. (Note also $S^2 = SO(3)/SO(2)$.)

Slow-fast decomposition:

$$Q(r) = u(r)\, e^{Y(r)}\, \sigma_3\, e^{-Y(r)}\, u(r)^{-1},$$

$u(r) \in SU(2)$ "slow",

$Y(r) = i(\sigma_1 y^1(r) + \sigma_2 y^2(r))$ "fast".

Notation (good for any symmetric space $M = G/K$):

$\underline{p} = \mathrm{span}_{\mathbb{R}}\{i\sigma_1, i\sigma_2\}$, $\underline{k} = \mathbb{R}\cdot i\sigma_3 = \mathrm{Lie}\,K$

$\mathrm{Lie}\,SU(2) = \underline{k} \oplus \underline{p}$ (orthogonal sum).

Commutation relations:

$$[\underline{p}, \underline{p}] \subset \underline{k}, \quad [\underline{p}, \underline{k}] \subset \underline{p}, \quad [\underline{k}, \underline{k}] \subset \underline{k} \ (\overset{\text{here}}{\equiv 0}).$$

Note that the factorization $g = u\, e^{Y}$ comes with a redundancy, which entails gauge invariance:

$$u(r) \longmapsto u(r)\, k(r)^{-1},$$
$$Y(r) \longmapsto k(r)\, Y(r)\, k(r)^{-1}, \qquad k(r) \in K \equiv U(1).$$

[Comment. This is closely related to thinking about the tangent bundle TS^2 as an associated vector bundle:

$TS^2 = SU(2) \times_{U(1)} \underline{p} \longrightarrow SU(2)/U(1)$; c.f. Sect. II.9]

<u>Trick</u> (to handle the metric in an efficient way):

Round metric of $S^2 = du_1^2 + du_2^2 + du_3^2 = \frac{1}{2} \mathrm{Tr}\,(dQ)^2$.

$$\mathrm{Tr}\,(dQ)^2 \overset{Q = g\sigma_3 g^{-1}}{=} \mathrm{Tr}\,(dg \cdot \sigma_3 g^{-1} - g\sigma_3 g^{-1} dg \cdot g^{-1})^2$$

$$= \mathrm{Tr}\,[g^{-1}dg, \sigma_3]^2 = -4\,\mathrm{Tr}\,(g^{-1}dg)_{\underline{p}}^2$$

where $X_{\underline{p}}$ denotes the projection of $X \in su(2)$ to $\underline{p}$.

Fix standard metric $\equiv - \mathrm{Tr}\,(g^{-1}dg)_{\underline{p}}^2$ (to suppress some constants).

Now, make preparation:

$$(g^{-1}dg)_{\underline{p}} = \left((u e^Y)^{-1} d(u e^Y)\right)_{\underline{p}}$$

$$= (u^{-1}du)_{\underline{p}} + dY - [Y, (u^{-1}du)_{\underline{k}}] + \frac{1}{2} \mathrm{ad}^2(Y)\,(u^{-1}du)_{\underline{p}} + O(Y^3).$$

Abbreviate $u^{-1}du \equiv X$. Then

$$(g^{-1}dg)_{\underline{p}} = X_{\underline{p}} + (d + \mathrm{ad}(X_{\underline{k}}))Y + \frac{1}{2}\mathrm{ad}^2(Y) X_{\underline{p}} + O(Y^3)$$

and $(-1)\cdot$ metric $= \mathrm{Tr}\,(g^{-1}dg)_{\underline{p}}^2 = A_0 + A_1 + A_2 + A_2' + \ldots$

where
$$A_0 = \mathrm{Tr}\, X_{\underline{p}}^2,$$
$$A_1 = 2\,\mathrm{Tr}\, X_{\underline{p}}\,(d + \mathrm{ad}(X_{\underline{k}}))Y,$$
$$A_2 = \mathrm{Tr}\,(dY + [X_{\underline{k}}, Y])^2,$$
$$A_2' = \mathrm{Tr}\, X_{\underline{p}}\, \mathrm{ad}^2(Y) X_{\underline{p}} = \mathrm{Tr}\, Y\, \mathrm{ad}^2(X_{\underline{p}}) Y.$$

Transcription to the field-theory action:

Evaluate u, Y on field map $\phi: \Sigma \to S^2$, so

$$u(r) \equiv u(\phi(r)), \quad Y(r) \equiv Y(\phi(r)).$$

① $$-A_0 = -\mathrm{Tr}\,(u^{-1}du)^2_{\underline{p}} \longrightarrow -\frac{1}{T}\int d^2r\, \mathrm{Tr}\,(u^{-1}\nabla u)^2_{\underline{p}}$$
$$= S[Q = u\sigma_3 u^{-1}].$$

② $$-A_1 = -2\,\mathrm{Tr}\, X_{\underline{p}}\,(d + \mathrm{ad}(X_{\underline{k}}))\,Y \longrightarrow$$
$$-\frac{2}{T}\int d^2r\, \mathrm{Tr}\,(u^{-1}\nabla u)_{\underline{p}} \cdot (\nabla Y + [(u^{-1}\nabla u)_{\underline{k}}, Y]).$$

Exercise. Argue that ② vanishes by the "equations of motion" $\frac{\delta}{\delta u} S = 0$ for the background field u. (This is the nonlinear generalization of the linear-case relation $\frac{\delta}{\delta \phi(r)}\Gamma = j(r) \equiv 0$.)

③ $$-A_2 - A_2' \longrightarrow$$
$$-\frac{1}{T}\int d^2r\, \mathrm{Tr}\left\{(\nabla Y + [(u^{-1}\nabla u)_{\underline{k}}, Y])^2 + Y\,\mathrm{ad}^2(u^{-1}\nabla u)_{\underline{p}}\,Y\right\}.$$

Remark (on video lecture). Very sorry, the term $[(u^{-1}\nabla u)_{\underline{k}}, Y]$ cannot be gauged away, unless $(u^{-1}\nabla u)_{\underline{k}}$ is a gradient field, which need not be true in general (if u varies in a non-Abelian group such as our $SU(2)$).

Integrate over the fast quantum field $Y(r)$:

Recall $Y(r) = i\left(\sigma_1 y^1(r) + \sigma_2 y^2(r)\right)$.

Let $\mathcal{H}^{(2)}_{ij} = \mathrm{Tr}\, \sigma_i \left(-\left(\nabla + \mathrm{ad}(u^{-1}\nabla u)_{\underline{k}}\right)^2 + \mathrm{ad}^2(u^{-1}\nabla u)_{\underline{p}}\right)\sigma_j$

Then $Z^{(1\text{-loop})} = \int \mathcal{D}y \; e^{-\frac{1}{T}\int d^2r \; y^i \mathcal{H}^{(2)}_{ij} y^j}$

$$= \text{const.} \, \mathrm{Det}^{-1/2}\left(\mathcal{H}^{(2)}\right).$$

! Some work still to be done ($\longrightarrow$ L25) to compute this functional determinant in order to arrive at the one-loop effective action. Final outcome

$$\frac{d}{d\ln a}\left(\frac{1}{T}\right) = -b + O(T) \; \checkmark$$

III.12 Computation of functional determinants by the heat-kernel method

Recall from III.11 the one-loop effective action of the O(3) nonlinear σ model:

$$\Gamma^{(1-\mathrm{loop})}[u\sigma_3 u^{-1}] = S[u\sigma_3 u^{-1}]$$
$$+\frac{1}{2}\operatorname{Tr}_{\underline{p}\times L^2(\mathbb{R}^2)} \ln\left\{-(\nabla + \mathrm{ad}(u^{-1}\nabla u)_{\underline{k}})^2 + \mathrm{ad}^2(u^{-1}\nabla u)_{\underline{p}}\right\}.$$

<u>Plan</u>: compute the gradient expansion of $\operatorname{Tr}\ln(\dots)$ by the so-called heat kernel method.

[<u>Comment</u>: gradient expansion is justified by the fact that terms of higher order in gradients are RG-irrelevant by power counting.]

Start from the basic identity (for λ a positive number)

$$\int_\varepsilon^\infty \frac{d\tau}{\tau} e^{-\lambda\tau} \overset{\varepsilon\to 0+}{=} -\ln(\varepsilon\lambda) + \underset{(\text{independent of } \lambda)}{\mathcal{O}(\varepsilon^0)} + \mathcal{O}(\varepsilon^1),$$

transcribed to the setting with an operator, say $D^2 > 0$:

$$\int_\varepsilon^\infty \frac{d\tau}{\tau} \operatorname{Tr} e^{-\tau D^2} = -\operatorname{Tr}\ln(\varepsilon D^2) + \underset{\text{independent of } D^2}{\text{const}} + \mathcal{O}(\varepsilon^1).$$

<u>Remark</u>. The name "heat kernel" (one also speaks of "proper time method") stems from the important example of $D^2 = -\Delta$ (Laplacian).

To take the trace, use (in physics-style notation)

$$\begin{aligned}
\mathrm{Tr}\, e^{-\tau D^2} &= \int d^dx \, \langle x | e^{-\tau D^2} | x \rangle \\
&= \int d^dx \; e^{-\tau D_x^2} \, \delta(x-x') \Big|_{x'=x} \\
&= \int d^dx \; e^{-\tau D_x^2} \int \frac{d^dk}{(2\pi)^d} \, e^{ik(x-x')} \Big|_{x'=x} \\
&= \int d^dx \int \frac{d^dk}{(2\pi)^d} \, e^{-ikx} \, e^{-\tau D^2} \, e^{+ikx} \\
&= \int d^dx \int \frac{d^dk}{(2\pi)^d} \, e^{-\tau D^2(\nabla \to \nabla + ik)} \cdot 1 \qquad \blacksquare
\end{aligned}$$

We have not yet regularized the functional determinant (or $\mathrm{Tr}\ln = \ln \mathrm{Det}$). For that, use Pauli–Villars regularization:

$$\mathrm{Det}(D^2) \to \mathrm{Det}_{PV}(D^2) \equiv \frac{\mathrm{Det}(D^2+m^2)}{\mathrm{Det}(D^2+M^2)} \quad \text{with}$$

small "mass" m^2 and large "mass" M^2 (= high-energy cutoff).

<u>Remark</u>. The insertion of a small mass term m^2 (for the purpose of infrared-regularization) is OK if the determinant is over the modes of a fast quantum field (<u>not</u> the slow background field).

A FORMULA:

$$\ln \mathrm{Det}_{PV}(D^2) = -\lim_{\varepsilon \to 0+} \int_\varepsilon^\infty \frac{d\tau}{\tau} \left(e^{-m^2\tau} - e^{-M^2\tau} \right) \mathrm{Tr}\, e^{-\tau D^2}$$

$$= -\lim_{\varepsilon \to 0} \int_\varepsilon^\infty \frac{d\tau}{\tau} \left(e^{-m^2\tau} - e^{-M^2\tau} \right) \int d^dx \int \frac{d^dk}{(2\pi)^d} \, e^{-\tau D^2(\nabla \to \nabla + ik)} \cdot 1 \qquad \blacksquare$$

Application of formula to our 2D case.

$$\Gamma^{(1-\text{loop})} - S =$$

$$-\frac{1}{2}\int d^2r \int_{\varepsilon\to 0}^{\infty} \frac{d\tau}{\tau}\left(e^{-m^2\tau} - e^{-M^2\tau}\right) \int \frac{d^2k}{(2\pi)^2} \operatorname{Tr}_{\underline{p}}\, e^{\tau\left(\nabla + ik + ad(X_{\underline{k}})\right)^2 - \tau\, ad^2(X_{\underline{p}})} \cdot 1$$

Now we would like to remove the linear operator $ad(X_{\underline{k}})$ by shifting the integration variable k. Doesn't work (at least not right away). However, let $A \equiv ad(X_{\underline{k}})$ and notice

$$e^{\tau(\nabla + ik + A)^2} \cdot 1 = e^{-k^2\tau}\, e^{\tau(\partial^\mu + A^\mu)(\partial_\mu + A_\mu + 2ik_\mu)} \cdot 1$$

$$= e^{-k^2\tau}\left\{1 + \tau(\partial^\mu + A^\mu)(A_\mu + 2ik_\mu) + \frac{\tau^2}{2}(\partial^\mu + A^\mu)\, 2ik_\mu \cdot A^\nu\, 2ik_\nu + \dots\right\}.$$

Now $\langle k_\mu \rangle \equiv \int \frac{d^2k}{(2\pi)^2} e^{-k^2\tau} k_\mu \Big/ \int \frac{d^2k}{(2\pi)} e^{-k^2\tau} = 0$

and $\langle k_\mu k_\nu \rangle = \frac{\delta_{\mu\nu}}{2\tau}$. So (cancellation of $O(\tau)$ terms)

$$\int \frac{d^2k}{(2\pi)^2} e^{\tau(\nabla + ik + A)^2} \cdot 1 = \frac{1}{4\pi\tau}\left(1 + O(\tau^2)\right).$$

Applying the differential operator $e^{\tau(\nabla + ik + A)^2}$ to

$e^{-\tau\, ad^2(X_{\underline{p}})}$ (instead of just 1) gives higher-order derivatives, which are RG-negligible. Thus we arrive at

$$\Gamma^{(1-\text{loop})} - S =$$

$$= -\frac{1}{2}\int d^2r \int_{\varepsilon\to 0}^{\infty} \frac{d\tau}{\tau}\left(e^{-m^2\tau} - e^{-M^2\tau}\right) \underbrace{\int \frac{d^2k}{(2\pi)^2} e^{-k^2\tau} \operatorname{Tr}_{\underline{p}}\, e^{-\tau\, ad^2(X_{\underline{p}})}}_{= \frac{1}{4\pi\tau}\operatorname{Tr}_{\underline{p}}\left(1 - \tau\cdot ad^2(X_{\underline{p}}) + O(\tau^2)\right)}$$

$$= \text{const} - \frac{1}{2}\int d^2r \int_{\varepsilon\to 0}^{\infty} \frac{d\tau}{\tau}\left(e^{-m^2\tau} - e^{-M^2\tau}\right)\left(-\frac{1}{4\pi}\right) \mathrm{Tr}_{\mathfrak{p}}\, \mathrm{ad}^2(X_{\mathfrak{p}}) + \ldots$$

$$= \text{const} + \frac{1}{8\pi}\int d^2r\, \mathrm{Tr}_{\mathfrak{p}}\, \mathrm{ad}^2(u^{-1}\nabla u)_{\mathfrak{p}} \cdot \underbrace{\lim_{\varepsilon\to 0} \int_{\varepsilon m^2}^{\varepsilon M^2} \frac{d\tau}{\tau} e^{-\tau}}_{=\ \ln(M^2/m^2)} .$$

Now $\mathrm{Tr}_{\mathfrak{p}}\, \mathrm{ad}^2(i\sigma_2) = \frac{1}{2}\mathrm{Tr}\, \sigma_1 [i\sigma_2, [i\sigma_2, \sigma_1]] = -4$
$= 2\,\mathrm{Tr}\,(i\sigma_2)^2,$

hence $\mathrm{Tr}_{\mathfrak{p}}\, \mathrm{ad}^2(u^{-1}\nabla u)_{\mathfrak{p}} = +2\,\mathrm{Tr}\,(u^{-1}\nabla u)_{\mathfrak{p}}^2 .$

Altogether,

$$\Gamma^{(1\text{-loop})} = \left(\frac{1}{T} - \frac{1}{4\pi}\ln(M^2/m^2)\right)\int d^2r\, (-\mathrm{Tr})\,(u^{-1}\nabla u)_{\mathfrak{p}}^2 .$$

As a generating function for low-energy observables the effective action Γ must be RG-invariant.

Thus, since $\ln(M^2/m^2) = -2\ln a + \text{const}$ $(a \equiv a_{uv})$

we have

$$0 = \frac{d}{d\ln a}\Gamma^{(1\text{-loop})} \wedge 0 = \frac{d}{d\ln a}\left(\frac{1}{T} + \frac{1}{2\pi}\ln a\right)$$

$$\text{or} \quad \frac{d}{d\ln a}\left(\frac{1}{T}\right) = -\frac{1}{2\pi} < 0 \; \blacksquare$$

<u>Info</u> (ε-expansion). Going away from $d=2$ (Friedan)

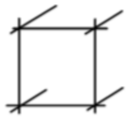
$$\frac{d}{d\ln a}\left(\frac{g_{ij}}{T}\right) = (d-2)\frac{g_{ij}}{T} - \mathrm{Ric}_{ij} - \frac{T}{2} R_{ipqr} R_j{}^{pqr} + \ldots$$

Expect RG-fixed point at $T \sim \varepsilon \equiv d-2$ for $\varepsilon > 0$ (and target with positive Ricci curvature) $\blacksquare$

Chapter C: Gauge Theories of Quantum Matter

Overview: Ising gauge theory & lattice gauge theory
gauge theories as effective field theories of low-energy response: topological quantum matter

C.1 Chains and cochains

Consider some lattice K (e.g. a cubic lattice: )

made from vertices = sites = 0-cells,
edges = links = 1-cells,
faces = plaquettes = 2-cells, etc.

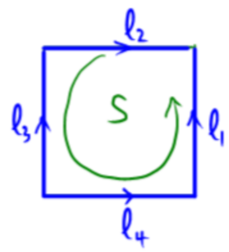
A k-chain n is a formal linear combination of k-cells: $n = \sum n_c \cdot c$ (coefficient n_c, k-cell c)

The k-chains on K form a vector space denoted by $C_k(K)$.

There exists a linear operator $\partial : C_k(K) \longrightarrow C_{k-1}(K)$, called the boundary operator.

Examples:

$$\partial \ell = (+1)\cdot p_f + (-1)\cdot p_i = p_f - p_i \,.$$



$$\partial S = \ell_1 - \ell_2 - \ell_3 + \ell_4$$

One has $\partial \circ \partial = 0$ (the boundary of a boundary always vanishes).

The elements θ of the dual vector space $C^k(K) := C_k(K)^*$ are called k-cochains.

They are linear combinations of linear functions on k-cells: $\theta = \sum \theta_c \cdot c^*$ (coefficient θ_c, dual k-cell c^*)

Dual basis: $c^*(c') = \delta_{cc'}$.

Pairing between C_k and C^k: $\langle n, \theta \rangle := \theta(n) = \sum_{cc'} \theta_c n_{c'}\, c^*(c') = \sum_c n_c \theta_c$

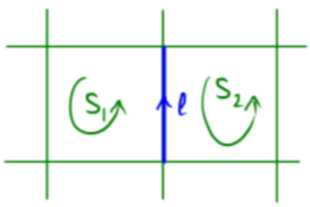
The coboundary operator $d : C^{k-1} \longrightarrow C^k$ is defined by the relation

$\langle n, d\theta \rangle = \langle \partial n, \theta \rangle$. Thus d is the adjoint (or transpose) of ∂.

It follows that $d \circ d = 0$.

Examples:

$$dp^* = -\ell_1^* - \ell_2^* + \ell_3^* + \ell_4^*$$



$$d\ell^* = S_1^* - S_2^*$$

C.2 Ising model and Ising gauge theory

We now introduce a large class of (statistical mechanics) models as follows.

A configuration of the field, say θ, is a k-cochain (on some lattice K with boundary operator ∂ and coboundary operator d), $\theta = \sum \theta_c \cdot c^*$, with coefficients θ_c that take values in an Abelian group G. The energy function H of the model is a functional of $d\theta$; typically $H(\theta) = \sum_\gamma h(d\theta_\gamma)$ (sum over all $(k+1)$-cells γ). The partition function is $Z = \int \mathcal{D}\theta \, e^{-\beta H(\theta)}$ with $\mathcal{D}\theta = \prod_c d\theta_c$ (product of Haar measures; of course, if G is discrete then the integral is a sum).

Example 1: $k=0$, $G = U(1)$ (xy model in D dimensions).

Realize the Abelian group $G = U(1)$ (where the composition law $G \times G \to G$ is multiplication $(z_1, z_2) \mapsto z_1 z_2$) as $G \cong \mathbb{R}/2\pi\mathbb{Z}$ (where the composition law is addition $(\theta_1, \theta_2) \mapsto \theta_1 + \theta_2$) by $z = e^{i\theta}$.

The field $\theta = \sum_s \theta_s \cdot s^*$ is a 0-cochain (or function) assigning to each 0-cell (or site) s an angle θ_s. We have $d\theta = \sum_\ell (\theta_{\varepsilon(\ell)} - \theta_{\alpha(\ell)}) \, \ell^*$ where $\varepsilon(\ell)$ and $\alpha(\ell)$ are the sites where the link ℓ ends resp. begins. If we choose $h(\phi) = -\cos\phi$ we get the so-called xy model (or planar model)

$$Z = \int \mathcal{D}\theta \; e^{\beta \sum \cos(\theta_s - \theta_{s'})}$$

where the sum is over nearest neighbor sites (on a lattice K in D dimensions).

Example 2: $k=0$, $G = \mathbb{Z}_2$ (Ising model)

The setting is as before but we now restrict the values of θ to $\{0, \pi\}$. This corresponds to $z \in \{\pm 1\}$ for $z = e^{i\theta}$. For the local energy h on links we take $h(0) = -J$ and $h(\pi) = +J$ $(J > 0)$. If we switch to the standard notation $s \equiv e^{i\theta} = \pm 1$ for spin values and i,j for lattice sites then we get

$$H = -J \sum_{\langle i,j \rangle} s_i \, s_j$$

which is the energy function of the Ising model.

Example 3: $k=1$, $G = U(1)$ ($U(1)$ lattice gauge theory)

Here a configuration of the field is a 1-cochain $\theta = \sum \theta_\ell \cdot \ell^*$ with $U(1)$-valued coefficients (we still realize G as $\mathbb{R}/2\pi\mathbb{Z}$). The field θ assigns an angle θ_ℓ to every link ℓ and is called an (Abelian) gauge field. The energy function H depends only on $d\theta$. Due to $d^2 = 0$, the energy H is invariant under gauge transformations $\theta \mapsto \theta + df$ where f is any 0-cochain. Usually one takes H to be a sum over plaquettes (or 2-cells):
$$H(\theta) = \sum_p h(d\theta_p).$$

Wilson form of h: $h_w(\phi) = -\cos\phi$,

Villain form: $h_V(\phi) = -\frac{1}{\beta} \ln\left(\sum_{m\in\mathbb{Z}} e^{-m^2/2\beta + im\phi} \right)$.

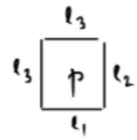
By the Poisson summation formula one has the alternative expression
$$h_V(\phi) = -\frac{1}{\beta}\ln\left(\sqrt{2\pi\beta}\sum_{n\in\mathbb{Z}} e^{-(\beta/2)(\phi+2\pi n)^2}\right) \overset{\beta\gg 1}{\approx} \text{const}_\beta + \frac{1}{2}\phi^2 \dots$$

Thus for low temperatures ($\beta \gg 1$) the Wilson and Villain forms of h are equivalent.

Example 4: $k=1$, $G = \mathbb{Z}_2$ (Ising gauge theory)

Again, we restrict to values $\theta_\ell \in \{0,\pi\} \longleftrightarrow s_\ell = e^{i\theta_\ell} \in \{\pm 1\}$.

Just like in the Ising spin model, we choose the local energy function $h(0) = -1$ and $h(\pi) = +1$ (without loss, we put $J=1$). In terms of Ising spin variables $s_\ell = e^{i\theta_\ell}$ the partition function is $Z = \sum e^{\beta\sum s_p}$ where $s_p = s_{\ell_1}s_{\ell_2}s_{\ell_3}s_{\ell_4}$ is the product of Ising spins for the four links bounding the plaquette p:



$\mathbb{Z}_2$ gauge invariance in this context means that the energy of a configuration does not change when one flips the Ising spin for all the links that emanate from a site.

C.3 Duality transformation

For any one of the class of models introduced in Section C.2 we now carry out a duality transformation of the Kramers–Wannier type (high $T \leftrightarrow$ low T).

The field is a k-cochain $\theta = \sum \theta_c \cdot c^*$ with coefficients $\theta_c \in \mathbb{R}/2\pi\mathbb{Z}$. We do the calculation for the Villain form of H (for the Wilson form it wouldn't be much different). Introducing the abbreviation $\|m\|^2 := \sum_\gamma m_\gamma^2$ for the sum of squares of the integer-valued coefficients m_γ of a $(k+1)$-chain $m = \sum_\gamma m_\gamma \cdot \gamma$, we write the partition function as

$$Z = \int \mathcal{D}\theta \; e^{-\beta H(\theta)} = \int \mathcal{D}\theta \sum_m e^{-\|m\|^2/2\beta + i\langle m, d\theta\rangle}.$$

The inner sum is over all $(k+1)$-chains m with coefficients in $\mathbb{Z}$. The pairing $\langle m, d\theta\rangle = \sum_\gamma m_\gamma (d\theta)_\gamma$ is defined modulo $2\pi\mathbb{Z}$, and therefore the exponential $e^{i\langle m, d\theta\rangle}$ is well-defined. We now use the defining relation $\langle m, d\theta\rangle = \langle \partial m, \theta\rangle$ of the coboundary operator d to express the partition function as

$$Z = \sum_m e^{-\|m\|^2/2\beta} \int \mathcal{D}\theta \; e^{i\langle \partial m, \theta\rangle}.$$

We have also interchanged the order of integration and summation. Now the integral $\int \mathcal{D}\theta \; e^{i\langle \partial m, \theta\rangle}$ vanishes unless m has zero boundary: $\partial m = 0$. Hence

$$Z = \sum_{m:\, \partial m = 0} e^{-\|m\|^2/2\beta}. \qquad \left(\int \mathcal{D}\theta = 1 \text{ by the choice of normalization for } \mathcal{D}\theta\right)$$

Recall that $\partial^2 = 0$, i.e. $\mathrm{im}\left(\partial : C_{k+2} \to C_{k+1}\right) \subseteq \ker\left(\partial : C_{k+1} \to C_k\right)$. We now assume the stronger property $\mathrm{im}\left(\partial : C_{k+2} \to C_{k+1}\right) = \ker\left(\partial : C_{k+1} \to C_k\right)$. (This is known as the Poincaré-Lemma or, more precisely, the vanishing of the homology group $H_{k+1}(K)$. This property holds true, for example, for a cubic lattice $K = \mathbb{Z}^D$.) We can then solve the constraint $\partial m = 0$ by setting $m = \partial n$. The partition function becomes a sum over $(k+2)$-chains n with coefficients in $\mathbb{Z}$:

$$Z = \text{const} \cdot \sum_n e^{-\|\partial n\|^2/2\beta}.$$

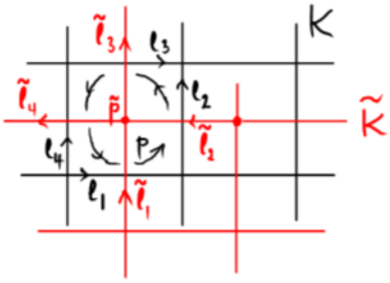
To be sure, the chain n is not uniquely determined by the equation $m = \partial n$. Indeed, a gauge transformation $n \mapsto n + \partial v$ (for any $(k+3)$-chain v) leaves m unchanged.

Therefore, the substitution $m = \partial n$ simply causes a change of overall normalization constant which is independent of n and β (although it may be infinite, in which case we have to "fix the gauge").

What we have achieved up to now is a reformulation of our original theory of k-cochains θ as a new theory of $(k+2)$-chains n. To complete the duality transformation, we make a conversion from chains back to cochains. This is achieved by passing from the lattice K to its dual lattice $\tilde{K}$. Given a D-dimensional lattice K, the dual lattice $\tilde{K}$ is defined by the requirement that there be a bijection $C_p(K) \overset{1:1}{\longleftrightarrow} C^{D-p}(\tilde{K})$ for all $p = 0, 1, \dots D$. This is implemented by a one-to-one correspondence between the p-cells of K and the $(D-p)$-cocells of $\tilde{K}$. The correspondence is such that the boundary operator $\partial : C_p(K) \longrightarrow C_{p-1}(K)$ corresponds (up to a sign) to the coboundary operator $\pm d : C^{D-p}(\tilde{K}) \longrightarrow C^{D-p+1}(\tilde{K})$.

<u>Example</u>: K = square lattice in 2 dimensions $\cong \tilde{K}$.

Each 1-cell ℓ of K together with its partner $\tilde{\ell}$ in $\tilde{K}$ is oriented according to the counterclockwise sense:



$$\partial p = \ell_1 + \ell_2 - \ell_3 - \ell_4$$

$$d\tilde{p}^* = \tilde{\ell}_1^* + \tilde{\ell}_2^* - \tilde{\ell}_3^* - \tilde{\ell}_4^*$$

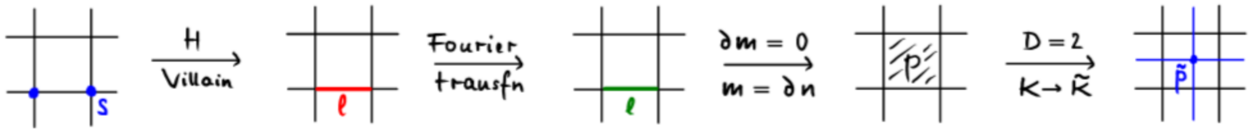
Returning to our general case, we have a bijection $n = \sum n_c \cdot c \longleftrightarrow \tilde{n} = \sum n_c \cdot \tilde{c}^*$ between $(k+2)$-chains n on K and $(D-k-2)$-cochains on $\tilde{K}$ such that ∂n corresponds to $\pm d\tilde{n}$. Thus we obtain the following final result for our duality transformation:

$$Z = \int \mathcal{D}\theta \; e^{-\beta H(\theta)} = \text{const} \cdot \sum_{\tilde{n}} e^{-\| d\tilde{n} \|^2 / 2\beta} .$$

Recall that $H(\theta)$ depends quadratically on $d\theta$ for $\beta \gg 1$. Thus our duality transformation is a strong-coupling-to-weak-coupling duality taking high temperature to low temperature and vice versa.

Example: $D=2$, $k=0$, K = square lattice

We recapitulate graphically the various steps of the duality transformation for the particular case at hand.



$\theta = \sum \theta_s \cdot s^*$ $\quad$ $d\theta = \sum (d\theta)_\ell \cdot \ell^*$ $\quad$ $m = \sum m_\ell \cdot \ell$ $\quad$ $n = \sum n_p \cdot p$ $\quad$ $\tilde{n} = \sum n_p \cdot \tilde{p}^*$

Now we further specialize to the case of $G = \mathbb{Z}_2$ (Ising spins). We realize the Ising spins on K as $\sigma_s = e^{i\theta_s}$ with $\theta_s \in \{0, \pi\}$ and the Ising spins on the dual lattice $\tilde{K}$ as $\sigma_{\tilde{p}} = e^{i\pi n_p}$ with $n_p \in \{0,1\}$. The previous calculation still goes through. Thus our duality transformation takes the 2D Ising model into itself, albeit with a change of coupling (or temperature) $\beta \longmapsto f(\beta)$.

The statistical weights before and after the duality transformation are as follows:

$$\left.\frac{P(\sigma=-1)}{P(\sigma=+1)}\right|_{\text{before}} = \frac{\sum_{m=0,1} e^{-m^2/2\beta}(-1)^m}{\sum_{m=0,1} e^{-m^2/2\beta}(+1)^m} = \frac{1 - e^{-1/2\beta}}{1 + e^{-1/2\beta}} = \tanh(1/4\beta),$$

$$\left.\frac{P(\sigma=-1)}{P(\sigma=+1)}\right|_{\text{after}} = \frac{e^{-m^2/2\beta}\big|_{m=1}}{e^{-m^2/2\beta}\big|_{m=0}} = e^{-1/2\beta} \equiv \tanh(1/4\beta'), \text{ hence}$$

$$\beta' = f(\beta) = \frac{1}{4}\left(\operatorname{Artanh}\left(e^{-1/2\beta}\right)\right)^{-1}.$$

(Here we are comparing Ising spins $\sigma_\ell = \sigma_{s_1}\sigma_{s_2} = \pm 1$ on links.)

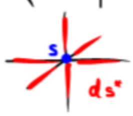
Notice that $f(0) = \infty$ and $f(\infty) = 0$. It is known that the 2D Ising model has a phase transition at a critical (inverse) temperature β_c. The duality transformation with change of coupling $\beta \longmapsto f(\beta)$ exchanges the high-temperature (disordered) phase with the low-temperature (ordered) phase. The critical temperature is determined by the fixed-point condition $\beta_c = f(\beta_c)$.

Example: $D=3$, $k=0$, K = cubic lattice.

The calculation is the same as in the previous example but for the final step. 
There, because of the increase in dimension ($D = 2 \to 3$), the passage from the lattice K to the dual lattice $\tilde{K}$ converts the 2-chain n to a 1-cochain $\tilde{n}$.
Let $G = \mathbb{R}/2\pi\mathbb{Z}$, as before. The initial model then is the xy-model in 3D, and our duality transformation takes it into a 3D gauge theory of $\mathbb{Z}$-valued fields with gauge group $\hat{G} = \mathbb{Z}$. A special situation arises for $G = \mathbb{Z}_2 = \hat{G}$.
In this case we learn that the 3D Ising model is dual to the 3D Ising gauge theory.
Since the 3D Ising model has a phase transition, it follows that so does the 3D Ising gauge theory. It turns out that the ordered (disordered) phase of the 3D Ising model corresponds (by duality) to a confinement (resp. deconfinement) phase of the 3D Ising gauge theory. (Some aspects of the latter will be discussed in the next section.)

C.4 Wegner–Wilson loop

We have just learned that the Ising gauge theory undergoes a phase transition in 3 dimensions. The existence of this transition raises the question what to use as a diagnostic for it (and also analogous phase transitions in other gauge theories).

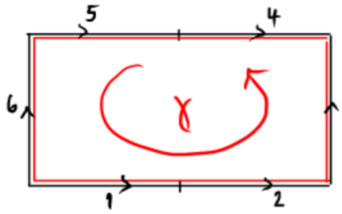
For concreteness of notation, we will give the answer for the example of $U(1)$ gauge theory. (As before, the Ising case will follow by a simple transcription.)
We begin by communicating that the conventional strategy for spin systems does not work here. As we already know, for a theory of planar spins ($k=0$ cochains) the high- and low-temperature phases are distinguished by $\langle e^{i\theta_s}\rangle = 0$ ($\beta < \beta_c$, disordered phase) and $\langle e^{i\theta_s}\rangle \neq 0$ (spontaneous breaking of $U(1)$-symmetry for $\beta > \beta_c$; ordered phase). There is no analog of such a diagnostic for gauge theories. The reason is that the "global" symmetry $\theta_s \mapsto \theta_s + \text{const}$ of a system of planar spins (or $\sigma_j \mapsto -\sigma_j$ in the Ising case) becomes a 'local' symmetry transformation $\theta \mapsto \theta + d\chi$ for $k \geq 1$ (here θ and χ are cochains of degree k and $k-1$ respectively). For example, for $k=1$ we may take $\chi = \phi \cdot s^*$ (for any site s). Then $d\chi = \phi \cdot ds^*$ involves only the links adjacent to s: 

. Now, guided by the procedure for spin systems, one might break the gauge invariance of $H(\theta) = H(\theta + d\chi)$ and consider a gauge-invariant observable in the limit of vanishing symmetry-breaking parameter.
However, this limit is uninformative because it always gives zero — a result known as Elitzur's Theorem ($\leftrightarrow$ the local gauge symmetry of a gauge theory cannot be spontaneously broken). In fact, no loss of interchangability of limits can occur here, as only a finite number of degrees of freedom is affected by a local gauge transformation.

The good diagnostic to use is a gauge-invariant observable called the Wegner–Wilson loop: pick an oriented loop γ (i.e. a closed directed curve γ) composed of links. (The simplest example would be the links running around an elementary plaquette.) Let the Abelian gauge group $G = \mathbb{Z}_2$, $U(1)$, etc. be implemented additively ($g = e^{i\theta}$). Viewing γ as a 1-cochain with coefficients in $\mathbb{Z}$ and using the pairing $\langle \cdot , \cdot \rangle$ between 1-chains and 1-cochains define the expectation value (Wegner 197•, Wilson 197•)

$$W(\gamma) := \left\langle e^{i\langle \gamma, \theta\rangle} \right\rangle .$$

Examples of $e^{i\langle\gamma,\theta\rangle}$



$$G = U(1): \quad e^{i\langle\gamma,\theta\rangle} = e^{i(\theta_1+\theta_2+\theta_3-\theta_4-\theta_5-\theta_6)} = g_1\, g_2\, g_3\, g_4^{-1}\, g_5^{-1}\, g_6^{-1}$$

$$G = \mathbb{Z}_2 : \quad e^{i\langle\gamma,\theta\rangle} = s_1\, s_2\, s_3\, s_4\, s_5\, s_6$$

Note that $e^{i\langle\gamma,\theta\rangle}$ is invariant under gauge transformations $\theta \mapsto \theta + d\alpha$ since γ is closed (i.e. $\partial\gamma = 0$). Indeed, $e^{i\langle\gamma,\theta\rangle} \longmapsto e^{i\langle\gamma,\theta+d\alpha\rangle} = e^{i\langle\gamma,\theta\rangle + i\langle\partial\gamma,\alpha\rangle} = e^{i\langle\gamma,\theta\rangle}$.

If $\gamma = \partial\Sigma$ then $e^{i\langle\gamma,\theta\rangle}$ can be put in a manifestly gauge-invariant form:

$$e^{i\langle\gamma,\theta\rangle} = e^{i\langle\partial\Sigma,\theta\rangle} = e^{i\langle\Sigma, d\theta\rangle}.$$

Interpretation. If θ is given the physical dimension of a gauge field, $[\theta] = \frac{\text{action}}{\text{el. charge}}$, then the pairing $\langle\gamma,\theta\rangle \in \mathbb{R}$ requires that $[\gamma] =$ electric charge. One interprets γ as the (closed) world line of a charge-anticharge pair. $-\ln W(\gamma)$ measures the change in gauge-field action (energy $\times$ time) due to the presence of the pair.

FACT (Wegner 197•). The asymptotics of $W(\gamma)$ for large γ is different in the two different phases of 3D Ising gauge theory:

$$W(\gamma) \sim \begin{cases} e^{-a(T)\cdot \text{area}(\gamma)} & \text{area law for high temperature } T > T_c, \\ e^{-b(T)\cdot \text{length}(\gamma)} & \text{perimeter law for low temperature } T < T_c. \end{cases}$$

Before we discuss this further, we make a generalization/unification.

Vocabulary: A k-chain c is called a $\begin{cases} k\text{-cycle if } \partial c = 0 & (c \in Z_k), \\ k\text{-boundary if } c = \partial b & (c \in B_k). \end{cases}$

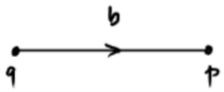
A k-cochain ω is called a $\begin{cases} k\text{-cocycle if } d\omega = 0 & (\omega \in Z^k), \\ k\text{-coboundary if } \omega = d\theta & (\omega \in B^k). \end{cases}$

Note that $B_k \subset Z_k$ and $B^k \subset Z^k$.

Now, given a statistical mechanics system of k-cochains θ with gauge-invariant energy function $H(\theta) = H(\theta + d\alpha)$ we can consider for any k-boundary $c = \partial b$ the generalized "Wegner-Wilson loop" $\quad W(c) := \left\langle e^{i\langle c,\theta\rangle}\right\rangle$.

Note that $e^{i\langle c,\theta\rangle}$ is invariant under gauge transformations as long as c is a k-cycle.

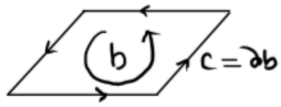
Examples:

$k=0$: 

$$e^{i\langle c,\theta\rangle} = e^{i\theta(p)-i\theta(q)} \equiv e^{i\int_q^p d\theta}$$

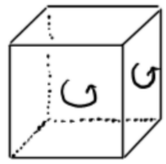
$c = p - q$

Here $W(c) = \left\langle e^{i\theta(p)}\, e^{-i\theta(q)} \right\rangle$ is the spin-spin correlation function.

$k=1$: 

$$e^{i\langle c,\theta\rangle} \equiv e^{i\int_c \theta} = e^{i\iint_b d\theta}$$

$W(c) =$ standard WWL

$k=2$: 

$c = \partial b$
surface of a right-handed cube

$$e^{i\langle c,\theta\rangle} \equiv e^{i\iint_c \theta} = e^{i\iiint_b d\theta}$$

We will now show that the (generalized) area law for $W(c)$ emerges naturally from a High-temperature (or strong-coupling) expansion:

Assuming the Villain form of the energy function we pass to the dual description:
(k - cochain θ)

$$W(c) = \frac{1}{Z}\int \mathcal{D}\theta\; e^{i\langle c,\theta\rangle} \sum_m e^{-\|m\|^2/2\beta \,+\, i\langle m, d\theta\rangle}$$

$$= \frac{1}{Z}\sum_m e^{-\|m\|^2/2\beta} \int \mathcal{D}\theta\; e^{i\langle c+\partial m,\theta\rangle}$$

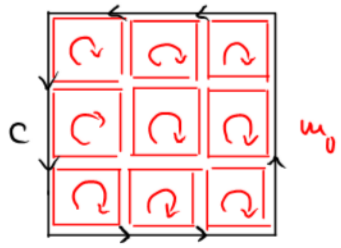
$$= \frac{1}{Z}\sum_{m:\,\partial m=-c} e^{-\|m\|^2/2\beta} = \sum_{m:\,\partial m=-c} e^{-\|m\|^2/2\beta} \Big/ \sum_{m:\,\partial m=0} e^{-\|m\|^2/2\beta} .$$

This reformulation is useful for small β (or high temperature), in which case only a small number of terms contribute to the sum over m. For very small β the partition sum is saturated by the trivial configuration $m=0$, and the numerator is saturated by the configuration m that minimizes $\|m\|^2$ under the constraint of $\partial m = -c$. Some examples of minimal chains m_0 (subject to $\partial m_0 = -c$) are:

$k=0$: $c = p - q$ 

$$W(c) = e^{-\,\mathrm{length}(m_0)/2\beta}$$

(spin-spin correlation function falls off exponentially with distance)

$k=1$: 

$$W(c) = e^{-\,\mathrm{area}(m_0)/2\beta} \qquad \text{(area law).}$$

As β increases (or the temperature decreases), more and more terms start contributing to the sum over m. For large c the generalized area law continues to hold (albeit with a renormalized coefficient or "string tension") up to the critical point β_c, where the coefficient of the area term goes to zero. (The fluctuations of the $(k+1)$-chain m become large, and eventually the hyper-surface m gets delocalized.)

Interpretation. Recall the physical meaning of c (for $k=1$) as a charge-anticharge loop in spacetime. If the loop is rectangular with sides R (= distance between charge and anticharge) and T (= imaginary time for which the pair is imposed on the vacuum), then the area law $-\ln W(c) \propto RT$ implies a charge-anticharge potential that grows linearly with R. Thus the area law signals confinement (of charges).

On the other hand, for $\beta > \beta_c$ (and $k=1$, $d=3$) the Wegner-Wilson loop of the Ising gauge theory is known to obey a perimeter law. (This can be seen by making a high-temperature expansion in the dual Ising spin theory.) check! In Section C.6 below we will look into the possibility for such a scenario to occur in the case of $k=1$, $d=3$, $G=U(1)$.

C.5 Laplacian on k-cochains

For the purpose of doing a low temperature expansion, we inject a mathematical intermezzo to introduce the lattice Laplacian on k-cochains ($k \geq 0$). For $k=1$ and $d=3$ this will be a lattice analog of the operator $\Delta = -\,\mathrm{rot}\circ\mathrm{rot} + \mathrm{grad}\circ\mathrm{div}$ on vector fields.

Reminder: The canonical pairing between chains and cochains gives us a canonical adjoint for the boundary operator ∂; that's the coboundary operator d:

$$\begin{tikzcd} C_k & C_{k+1} \arrow[l, "\partial"'] \\ \otimes & \otimes \\ C^k \arrow[r, "d"] \arrow[d] & C^{k+1} \arrow[d] \\ \mathbb{R} & \mathbb{R} \end{tikzcd}$$

$$\langle c, d\omega \rangle = \langle \partial c, \omega \rangle \quad \text{or} \quad \int_c d\omega = \int_{\partial c} \omega .$$

This is canonical in that it does not involve any quadratic form or metric.

Now let there be a non-degenerate symmetric bilinear form on chains, say

$$Q : C_k \times C_k \to \mathbb{R}, \quad Q(m,n) = \sum_c m_c n_c , \qquad \text{for all } k.$$

Then we have isomorphisms $\quad I : C_k \to C^k, \; n \mapsto Q(n,\cdot) \quad$ (for all k).

We then get an operator $d^\dagger : C^{k+1} \longrightarrow C^k$ for each k ($0 \leq k < D$) by the following commutative diagram:

$$\begin{tikzcd} C_k \arrow[d, "I"'] & C_{k+1} \arrow[l, "\partial"'] \\ C^k & C^{k+1} \arrow[l, "d^\dagger"] \arrow[u, "I^{-1}"'] \end{tikzcd}$$

$$d^\dagger = I \circ \partial \circ I^{-1}$$

The notation $d^\dagger$ is motivated by the following calculation:

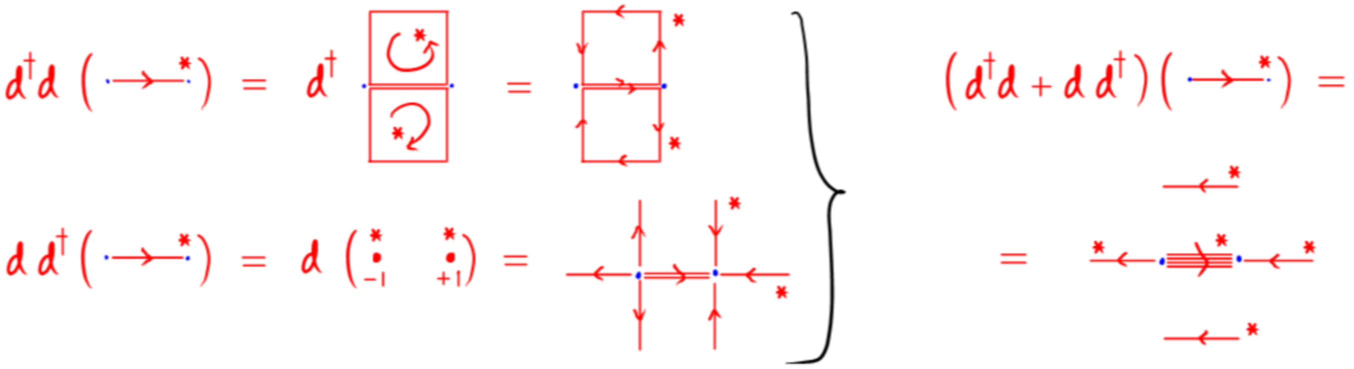
$$\langle m, \mathcal{I}(n)\rangle = Q(m,n) \rightsquigarrow \tilde{Q}(\alpha,\beta) := \langle \mathcal{I}^{-1}(\alpha), \beta\rangle , \quad \tilde{Q}(\alpha,\beta) = Q(\mathcal{I}^{-1}(\alpha), \mathcal{I}^{-1}(\beta)).$$

$$\tilde{Q}(\alpha, d\beta) = \langle \mathcal{I}^{-1}(\alpha), d\beta\rangle = \langle \partial \mathcal{I}^{-1}(\alpha), \beta\rangle = \langle \mathcal{I}^{-1}(\beta), \mathcal{I}\partial\mathcal{I}^{-1}(\alpha)\rangle = \tilde{Q}(\beta, \mathcal{I}\partial\mathcal{I}^{-1}(\alpha)).$$

Hence $\tilde{Q}(\alpha, d\beta) = \tilde{Q}(\beta, d^\dagger\alpha)$, which shows that $d^\dagger$ is adjoint to d by the quadratic form $\tilde{Q}$.

Given $d^\dagger$, the Laplacian on k-cochains is defined by $-\Delta = d^\dagger d + d d^\dagger$.

This operator has all the properties expected of a Laplacian. For example,



C.6 U(1) lattice gauge theory (3D) at large β

To deal with the low-temperature situation ($\beta > \beta_c$), we start from the expression for $W(c)$ of Sect. C.4 and solve the constraints $\partial m = 0$ and $\partial m = -c$ by setting $m = \partial n$ and $m = m_0 + \partial n$ (with $\partial m_0 = -c$) respectively:

$$W(c) = \sum_n e^{-\| m_0 + \partial n\|^2/2\beta} \Big/ \sum_n e^{-\|\partial n\|^2/2\beta} .$$

The sums are now over integer-valued 3-chains n, and the passage from m to n uses the Poincaré lemma ($H_2(K) = 0$ for K a cubic lattice).

The present formulation is no good in the limit of large β, as very many terms contribute to the sums over n. To arrive at a more suitable formulation, we first switch to the dual lattice:

$$W(c) = \sum_{\tilde{n}} e^{-\|\tilde{m}_0 + d\tilde{n}\|^2/2\beta} \Big/ \sum_{\tilde{n}} e^{-\|d\tilde{n}\|^2/2\beta} ,$$

where the sums are over 0-cochains $\tilde{n}$, and $\tilde{m}_0$ is the 1-cochain obtained by dualizing m_0. (Thus $d\tilde{m}_0 = -\tilde{c}$.) The key step now is to use Poisson summation in the (schematic) form
$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{q\in\mathbb{Z}} \int_{\mathbb{R}} d\phi \; f(\phi)\, e^{2\pi i q \phi} .$$

By applying this identity (at the lattice level) to both the numerator and the denominator of the expression for $W(c)$ we obtain

$$W(c) = \sum_q \int \mathcal{D}\phi \, e^{-\|\tilde{m}_0 + d\phi\|^2/2\beta \, + \, 2\pi i \langle q, \phi\rangle} \Big/ \sum_q \int \mathcal{D}\phi \, e^{-\|d\phi\|^2/2\beta \, + \, 2\pi i \langle q, \phi\rangle} .$$

Here the sums (integrals) are over $\mathbb{Z}$-valued 0-chains q ($\mathbb{R}$-valued 0-cochains ϕ).

<u>Interpretation.</u> We know from $\partial m_0 = -c$ that m has the physical dimension of electric charge. The same goes for its Poisson replacement ϕ. The physical meaning of ϕ is that of a magnetic scalar potential (in fact it is a potential for the Maxwell 1-form $*F = d\phi$). Thus we infer that q is the 0-chain of a magnetic charge distribution. The "atoms" of q are magnetic monopoles. ∎

The ϕ-integrals are Gaussian and can be carried out as follows:

$$-\frac{1}{2\beta}\|\tilde{m}_0 + d\phi\|^2 + 2\pi i \langle q, \phi\rangle = -\frac{1}{2\beta}\,\tilde{Q}(\tilde{m}_0 + d\phi, \tilde{m}_0 + d\phi) + 2\pi i\, \tilde{Q}(\mathcal{I}(q), d\phi) .$$

Now $\quad \tilde{Q}(\tilde{m}_0 + d\phi, \tilde{m}_0 + d\phi) = \|\tilde{m}_0\|^2 + \tilde{Q}(\phi, -\Delta\phi) + 2\,\tilde{Q}(\phi, d^\dagger \tilde{m}_0)$.

Hence, $\quad -\frac{1}{2\beta}\|\tilde{m}_0 + d\phi\|^2 + 2\pi i \langle q, \phi\rangle =$

$$= -\frac{1}{2\beta}\,\tilde{Q}\Big(\phi + (-\Delta)^{-1}(d^\dagger \tilde{m}_0 - 2\pi i \beta \mathcal{I}(q)), (-\Delta)\big(\phi + (-\Delta)^{-1}(d^\dagger \tilde{m}_0 - 2\pi i \beta \mathcal{I}(q))\big)\Big)$$
$$-\frac{1}{2\beta}\,\tilde{Q}(\tilde{m}_0, \tilde{m}_0) + \frac{1}{2\beta}\,\tilde{Q}\big(d^\dagger \tilde{m}_0 - 2\pi i \beta \mathcal{I}(q), (-\Delta)^{-1}(d^\dagger \tilde{m}_0 - 2\pi i \beta \mathcal{I}(q))\big). \quad \equiv X$$

By using $\quad \tilde{Q}(\tilde{m}_0, \tilde{m}_0) - \tilde{Q}(d^\dagger \tilde{m}_0, (-\Delta)^{-1} d^\dagger \tilde{m}_0)$

$$= \tilde{Q}\big(\tilde{m}_0, (-\Delta)^{-1}((d^\dagger d + d d^\dagger) - d d^\dagger)\, \tilde{m}_0\big) = \tilde{Q}(d\tilde{m}_0, (-\Delta)^{-1} d\tilde{m}_0)$$

one reorganizes the second line as

$$X \equiv -\frac{1}{2\beta}\,\tilde{Q}(d\tilde{m}_0, (-\Delta)^{-1} d\tilde{m}_0) - 2\pi^2\beta\, \tilde{Q}(\mathcal{I}(q), (-\Delta)^{-1}\mathcal{I}(q)) - 2\pi i \langle q, (-\Delta)^{-1} d^\dagger \tilde{m}_0\rangle .$$

<u>Remark.</u> This result is compatible with the requirement of invariance under gauge transformations $\tilde{m}_0 \mapsto \tilde{m}_0 + df$ (for any $\mathbb{Z}$-valued 0-cochain f). For the third summand the invariance property folows from $\langle q, (-\Delta)^{-1} d^\dagger d f\rangle = \langle q, f\rangle$ and the (lattice) Dirac quantization condition $e^{-2\pi i \langle q, f\rangle} \in e^{2\pi i \mathbb{Z}} = 1$.

We finally carry out the ϕ-integrals to arrive at (recall $d\tilde{m}_0 = -\tilde{c}$)

$$W(c) = e^{-\frac{1}{2\beta} Q(c, (-\Delta)^{-1} c)} \sum_q e^{-2\pi^2\beta\, Q(q, (-\Delta)^{-1} q) - 2\pi i \langle q, (-\Delta)^{-1} d^\dagger \tilde{m}_0\rangle} \Big/ \sum_q e^{-2\pi^2 \beta\, Q(q, (-\Delta)^{-1} q)} .$$

<u>Problem.</u> The sums are over q that satisfy $\langle q, 1\rangle = 0$. Explain why!

Discussion. One might now hope (?!) that all configurations but $q=0$ can be neglected in the limit of large β. As a quick check whether this approximation is reasonable, let us inspect the consequences of assuming
$$W(c) \overset{?}{\approx} e^{-\frac{1}{2\beta} Q(c, (-\Delta)^{-1} c)} .$$

For a large loop c one may approximate the lattice propagator $(-\Delta)^{-1}$ by the continuum propagator (Coulomb potential). Thus
$$W(c) \overset{?}{\approx} e^{-\frac{1}{8\pi\beta a} \oint_c dx \oint_c dx' \, |x-x'|^{-1}} \qquad \text{(lattice constant } a\text{)}.$$
There is no serious difficulty from the (artificial) singularity at $x = x'$ (just use the exact lattice propagator to cure this UV problem). However, we are facing an IR problem: in order to get the (naively expected) perimeter law from the outer integral $\int_c dx = \text{length}(c)$, the inner integral $\oint_c dx' \, |x-x'|^{-1}$ would have to approach a finite limit for large c but it is actually logarithmically divergent.

Note: in $D=4$ dimensions it all makes sense:
$$W(c) \overset{\checkmark}{=} e^{-\frac{\text{const}}{\beta} \oint_c dx \oint_c dx' \, |x-x'|^{-2}} = e^{-b(\beta) \cdot \text{length}(c)}, \qquad b(\beta) = \frac{\text{const}}{\beta} \oint_c dx' \, |x-x'|^{-2} .$$
The perimeter law signals a Coulomb phase ($\leftrightarrow$ free photons & deconfined electric charges) for large β.

Resolution. In $D=3$ the $U(1)$ lattice gauge theory does not exhibit any perimeter law for $W(c)$! In fact, Polyakov (in Phys. Lett. 59B (1975) 82–84) argued that $W(c)$ obeys an area law ($\leftrightarrow$ confinement) for all β. The physical picture is that the presence of magnetic monopoles leads to confinement of electric flux by the "dual Meissner effect".

C.7 Boson–vortex duality

We now take a look at $k=0$, $D=3$, $G=U(1)$ using the same techniques as before. Physical motivation for this case comes from two-dimensional bosons which undergo a quantum phase transition between a superfluid phase and a Mott insulator phase.

The earlier expression for $W(c)$ continues to be valid ($\phi \equiv A$, $q \equiv j$):
$$W(c) = \frac{1}{Z} \sum_j \int \mathcal{D}A \, e^{-\| \tilde{m}_0 + dA \|^2 / 2\beta \; + \; 2\pi i \langle j, A \rangle},$$
albeit with several changes of meaning. We now have $c = b - a$, and $W(c) = \langle e^{i\theta(b)} \, e^{-i\theta(a)} \rangle$

is the spin-spin correlation function. The real-valued Gaussian field A (integer-valued field j) is a 1-chain (1-cochain). Gauge invariance ($A \to A + df$) stipulates that j be a 1-cycle ($\partial j = 0$), i.e. a configuration of loops. These loops have an interpretation as the world lines of vortices (in the $U(1)$ boson field θ).

Fixing the gauge by $d^\dagger A = 0$ (Coulomb gauge) we have

$$\| dA \|^2 = \widetilde{Q}(dA, dA) = \widetilde{Q}(A, d^\dagger d A) = \widetilde{Q}(A, dd^\dagger A + d^\dagger d A) = \widetilde{Q}(A, -\Delta A).$$

The rest of the calculation goes as before and still results in

$$W(c) = \frac{1}{Z}\, e^{-\frac{1}{2\beta} Q(c, (-\Delta)^{-1} c)} \sum_j e^{-2\pi^2 \beta\, Q(j, (-\Delta)^{-1} j) - 2\pi i \langle j, (-\Delta)^{-1} d^\dagger \tilde{m}_0 \rangle}.$$

For large values of β vortices are suppressed, and the spin-spin correlation function takes the form

$$W(c) = e^{-\frac{1}{2\beta} Q(c, (-\Delta)^{-1} c)} = e^{-\frac{1}{2\beta}(-\Delta)^{-1}(a,a) - \frac{1}{2\beta}(-\Delta)^{-1}(b,b) + \frac{1}{\beta}(-\Delta)^{-1}(a,b)}$$

Using $(-\Delta)^{-1}(a,b) \approx (4\pi\, |a-b|)^{-1}$ we see that $W(c) = \langle e^{i\theta(b)}\, e^{-i\theta(a)} \rangle \xrightarrow{|a-b| \to \infty} \text{const}$ in this case. This is the superfluid phase where $U(1)$ symmetry is spontaneously broken.

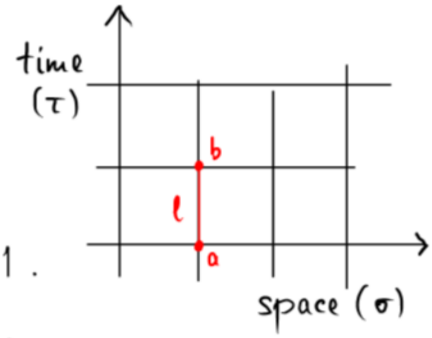
On the other hand, for β small vortices proliferate ($\leftrightarrow$ Mott insulator phase). As we have seen (Section C.4) the spin-spin correlation function falls off exponentially with distance in that case.

<u>Summary</u>. From Section C.3 we know that a D-dimensional theory of k-cochains with values in an Abelian group G is dual to a D-dimensional theory of $(D-k-2)$-cochains with values in the dual group $\widehat{G}$. To obtain from this general scenario the special case of boson-vortex duality, we set $D = 3$, $k = 0$, $G = U(1)$, which results in $k' = D - k - 2 = 1$ and $\widehat{G} = \mathbb{Z}$. The final representation by vortices emerges by using Poisson summation to separate the $\mathbb{Z}$-valued 1-cochain field of the dual side into an $\mathbb{R}$-valued 1-cochain (the gauge field A) paired with a $\mathbb{Z}$-valued 1-cycle (the vortex current j). If the Villain action is assumed for the original $U(1)$-boson theory, then the dual theory is Gaussian in the gauge field A and one can integrate out A exactly to produce a formulation in terms of Coulomb-interacting vortex current lines.

C.8 Hamiltonian formulation

Theme: statistical mechanics in D dimensions $\longleftrightarrow$

quantum theory in D−1 dimensions



Illustrate the main idea at the example of

$D = 2 = 1+1$, $k = 0$, $G = U(1)$:

$$Z = \int \mathcal{D}\theta \; e^{-\beta \sum_\ell V(d\theta_\ell)} ; \qquad V(0) = V'(0) = 0, \; V''(0) = 1 .$$

Anisotropic limit: (ε small)
$$\begin{cases} \beta \longrightarrow \beta_\tau := \beta/\varepsilon & \text{temporal links,} \\ \beta \longrightarrow \beta_\sigma := \beta\cdot\varepsilon & \text{spatial links .} \end{cases}$$

Temporal link ℓ ($\partial\ell = b - a$):

$$e^{-\beta_\tau V(d\theta_\ell)} \overset{\varepsilon\text{ small}}{\approx} e^{-\frac{\beta}{2\varepsilon}\left(\theta(b)-\theta(a)\right)^2} = \sqrt{2\pi\varepsilon/\beta}\; e^{\frac{\varepsilon}{2\beta}\frac{\partial^2}{\partial\theta^2}}\delta(\theta)\Big|_{\theta = \theta(b)-\theta(a)}$$

Spatial link ℓ ($\partial\ell = b - a$):

$$e^{-\beta_\sigma V(d\theta_\ell)} = e^{-\varepsilon\beta V\left(\theta(b)-\theta(a)\right)} .$$

Then $\quad Z = \mathrm{Tr}\left(e^{-\varepsilon\mathcal{H}}\right)^{N_\tau}, \qquad \mathcal{H} = -\frac{1}{2\beta}\sum_{\text{sites } s}\frac{\partial^2}{\partial\theta_s^2} + \beta\sum_{\ell:\,\partial\ell = s-s'} V(\theta_s - \theta_{s'}) + \text{const.}$

<u>Remark.</u> For $k=0$, $G = \mathbb{Z}_2$ (Ising model) the same procedure with anisotropy $\beta_\tau = \frac{1}{2}\ln\frac{\beta}{\varepsilon}$, $\beta_\sigma = \beta\varepsilon$ ($\varepsilon \to 0+$) yields the Hamiltonian $\mathcal{H}$ (acting on $(\mathbb{C}^2)^{\otimes N_\sigma}$) of the "transverse−field Ising model":

$$\mathcal{H} = -\frac{1}{\beta}\sum_{\text{sites } s}\hat{\sigma}^x_s - \beta\sum_{\ell:\,\partial\ell = s-s'}\hat{\sigma}^z_s\hat{\sigma}^z_{s'} .$$

For $k=1$ one uses the trick of gauging the field to zero on temporal links ("temporal gauge"). The same calculations (using the appropriate anisotropic limits) then give

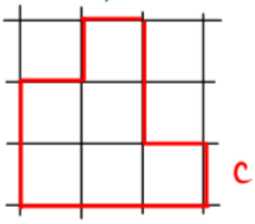
$$G = U(1): \quad \mathcal{H} = \underbrace{-\frac{1}{2\beta}\sum_{\text{links } \ell}\frac{\partial^2}{\partial\theta_\ell^2}}_{\text{electric energy}} + \underbrace{\beta\sum_{\text{plaqs } p} V(d\theta_p)}_{\text{magnetic energy}}$$

$$G = \mathbb{Z}_2: \quad \mathcal{H} = -\frac{1}{\beta}\sum_{\text{links } \ell}\hat{\sigma}^x_\ell - \beta\sum_{\text{plaqs } p}\hat{\sigma}^z_{\ell_1(p)}\hat{\sigma}^z_{\ell_2(p)}\hat{\sigma}^z_{\ell_3(p)}\hat{\sigma}^z_{\ell_4(p)}$$

C.9 Toric Code

In the area of topological quantum computing there exists a paradigmatic model, the "Toric Code" of A. Kitaev. The Toric Code Hamiltonian is a variant of the quantum Hamiltonian of the $(2+1)$-dimensional Ising gauge theory. It has the special feature that its electric and magnetic parts commute.

Setting. Let Λ be a two-dimensional lattice (i.e. a differential complex $C_2(\Lambda) \xrightarrow{\partial} C_1(\Lambda) \xrightarrow{\partial} C_0(\Lambda)$) with dual lattice $\tilde{\Lambda}$. There exists a canonical isomorphism $I: C_1(\Lambda) \longrightarrow C^1(\tilde{\Lambda})$; in particular, I takes cycles to cocycles and boundaries to coboundaries. Given $\Lambda, \tilde{\Lambda}$ one introduces two types of operator:



1) "Magnetic" loop operators (Wegner–Wilson):
For any cycle $c \in Z_1(\Lambda)$ let $B(c) := \prod_{\ell:\, \ell \in c} \hat{\sigma}_\ell^z$.

2) "Electric" loop operators:
For any cycle $\tilde{c} \in Z_1(\tilde{\Lambda})$ let $A(\tilde{c}) := \prod_{\ell:\, \langle \tilde{c}, I(\ell)\rangle \neq 0} \hat{\sigma}_\ell^x$

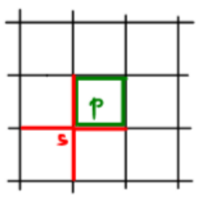
It is clear that all the operators $B(c)$ commute amongst themselves, and so do the $A(\tilde{c})$ (still amongst themselves). By the Clifford algebra relations obeyed by the Pauli matrices $\hat{\sigma}^z, \hat{\sigma}^x$, the commutation relations between the A's and B's are

$$A(\tilde{c})\, B(c) = (-1)^{\langle \tilde{c}, I(c)\rangle}\, B(c)\, A(\tilde{c}).$$

From $\langle Z_1, B' \rangle = 0 = \langle B_1, Z' \rangle$ (a boundary/coboundary pairs to zero with cocyle/cycle) it follows that $A(\tilde{c})$ and $B(c)$ commute with each other if at least one of $c, \tilde{c}$ is a boundary.

Hamiltonian: $\mathcal{H} = \mathcal{H}_{elec} + \mathcal{H}_{magn} = -J_e \sum_{\text{plaqs } \tilde{p}} A(\partial \tilde{p}) - J_m \sum_{\text{plaqs } p} B(\partial p)$

square lattice: $\mathcal{H} = -J_e \sum_{\text{sites } s} \hat{\sigma}^x_{\ell_1(s)} \hat{\sigma}^x_{\ell_2(s)} \hat{\sigma}^x_{\ell_3(s)} \hat{\sigma}^x_{\ell_4(s)} - J_m \sum_p \hat{\sigma}^z_{\ell_1(p)} \hat{\sigma}^z_{\ell_2(p)} \hat{\sigma}^z_{\ell_3(p)} \hat{\sigma}^z_{\ell_4(p)}$



Since ∂p and $\partial \tilde{p}$ are boundaries, we have

$$\left[\mathcal{H}_{elec}, \mathcal{H}_{magn}\right] = 0.$$

Thus a ground state of $\mathcal{H}$ is a simultaneous eigenstate of both $\mathcal{H}_{elec}$ and $\mathcal{H}_{magn}$. In fact, one can say a lot more:

<u>Fact.</u> For a lattice Λ with non-trivial homology, $C_1(\Lambda) \cap \ker \partial \,/\, C_1(\Lambda) \cap \operatorname{im} \partial \equiv H_1(\Lambda) \neq 0$, there are $2^{\dim H_1(\Lambda)}$ degenerate ground states.

<u>Sketch of proof.</u> For any 1-cycle c the Wegner–Wilson loop operator $B(c)$ commutes with $\mathcal{H}_{magn}$ and $\mathcal{H}_{elec}$ and hence with $\mathcal{H}$. Thus we may seek joint eigenstates of $\mathcal{H}$ and all operators $B(c)$. If c is a boundary ($c = \partial\Sigma$) then the eigenvalue of $B(c)$ on any ground state is $+1$. However, the eigenvalue may be -1 if c is a non-trivial cycle (not a boundary). This observation divides the ground states into $2^{\dim H_1(\Lambda)}$ distinct subspaces (by homology classes). None of these subspaces is empty (i.e. trivial). Indeed, we may apply an electric loop operator $A(\tilde{c})$ with $\langle \tilde{c}, I(c) \rangle \in 2\mathbb{Z}+1$ to flip the eigenvalue of $B(c)$.
One can show that the action of $\mathcal{H}$ is "ergodic" on every subspace given by homology. Thus each such subspace contains exactly one ground state. $\blacksquare$

C.10 Nonabelian gauge theory

Compact Lie group G; Lie algebra $\mathrm{Lie}(G)$.

Example: $G = SO(3)$; $\mathrm{Lie}\, SO(3) = \mathrm{span}_{\mathbb{R}} \{ J_x, J_y, J_z \}$

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Gauge field A is a vector field (actually, a 1-form) on space-time with values in $\mathrm{Lie}(G)$:

$$A_\mu(x,t) = \sum_a A^a_\mu(x,t)\, \tau_a \quad (\tau_a \text{ basis of } \mathrm{Lie}(G); \text{ normalization: } \mathrm{Tr}(\tau_a \tau_b) = -2\delta_{ab})$$

Field strength: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = [\partial_\mu + A_\mu, \partial_\nu + A_\nu]$

has the geometric meaning of a "curvature".

Recall from Riemannian geometry the definition of the Riemann curvature tensor:

Christoffel symbols

Levi-Civitá connection: $\nabla_\mu \partial_\nu = \Gamma^\lambda_{\mu\nu} \partial_\lambda$ or equivalently $\nabla_\mu = \partial_\mu + \Gamma_\mu$, $\Gamma_\mu = (\Gamma^\lambda_{\mu\nu})_{\nu\lambda}$.

Riemann curvature: $R_{\mu\nu} = [\partial_\mu + \Gamma_\mu, \partial_\nu + \Gamma_\nu] = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu]$.

$R_{\mu\nu}$ takes values in $\mathrm{Lie}\, SO(n)$ where $SO(n)$ is the orthogonal group of the tangent spaces $T_x M \simeq \mathbb{R}^n$ of the Riemannian manifold M.

While the Riemann curvature tensor R describes the (infinitesimal) holonomy determined by parallel transport of tangent vectors, the field-strength tensor F does the same for wave functions ψ (of particles with charges that couple to the non-Abelian gauge field A), as follows.

Let $[0,1] \ni s \mapsto \gamma(s)$ be a curve in space-time and set $\psi_s \equiv \psi(\gamma(s))$,

$A_s \equiv A_\mu(\gamma(s)) \frac{d}{ds} \gamma^\mu(s)$. Given any initial condition $\psi_0 = \psi(\gamma(0))$, solve the differential equation $(\partial_s + A_s)\psi_s = 0$ ("parallel" transport).

If the curve is closed $(\gamma(1) = \gamma(0))$ then $\psi_{s=1} = g \cdot \psi_{s=0}$

where $g \in G$ is called the holonomy associated with γ by A.

PICTURE

If the closed curve γ runs around a small square of area ε^2 in the $\mu\nu$-plane, then the holonomy is $g = 1 + \varepsilon^2 F_{\mu\nu} + \ldots$ $(\varepsilon \to 0)$.

$$F_{\mu\nu}(x,t) = \sum_a F^a_{\mu\nu}(x,t)\, \tau_a, \qquad F^a_{\mu\nu} = \begin{pmatrix} 0 & -E^a_x & -E^a_y & -E^a_z \\ E^a_x & 0 & B^a_z & -B^a_y \\ E^a_y & -B^a_z & 0 & B^a_x \\ E^a_z & B^a_y & -B^a_x & 0 \end{pmatrix}.$$

Gauge transformations are given by $\partial_\mu + A_\mu \mapsto g^{-1}(\partial_\mu + A_\mu)\circ g$ for $g(x,t) \in G$.

The induced transformation law for A is $A_\mu \mapsto g^{-1}(\partial_\mu + A_\mu)g = g^{-1}A_\mu g + g^{-1}\partial_\mu g$.

The field strength tensor is not gauge-invariant but transforms simply by conjugation: $F_{\mu\nu} \mapsto g^{-1}F_{\mu\nu}\, g$. To get gauge-invariant quantities one needs to take traces such as $\mathrm{Tr}\, F^{\mu\nu}F_{\mu\nu} = -2\sum_a F^{\mu\nu,a}F^a_{\mu\nu}$. The gauge-invariant action functional (of what is called Yang-Mills theory with gauge group G) is

$$S_{YM} = \frac{1}{4}\int d^3x\,dt\ \mathrm{Tr}\, F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}\int d^3x\,dt \sum_a F^{\mu\nu,a}F^a_{\mu\nu} = \int d^3x\,dt \sum_{a,i}\left(E_i^a E_i^a - B_i^a B_i^a\right)$$

To connect with our discussion of the Abelian theory on the lattice, we mention that there exists a non-Abelian analog of the Wegner-Wilson loop as follows. Similar to before (in the discussion of parallel transport & holonomy) let $[0,1] \ni s \mapsto \gamma(s)$ be a closed curve in space-time and define g_s by $g_{s=0} = \mathrm{Id}$ and solving the differential equation $(\partial_s + A_s)\, g_s = 0$ for a given gauge-field configuration A. The trace $\mathrm{Tr}\, g_{s=1}$ of the holonomy is gauge-invariant. Symbolically one writes

$$\mathrm{Tr}\, g_{s=1} = \mathrm{Tr}\, \mathcal{P}\, e^{-\oint_\gamma A} \quad \text{(Wegner-Wilson loop observable).}$$

Of particular interest for the following is the "topological" action

$$S_{top} = \frac{1}{4}\int d^3x\,dt\ \epsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\, F_{\mu\nu}F_{\lambda\rho} = -\frac{1}{2}\int d^3x\,dt\ \epsilon^{\mu\nu\lambda\rho} F^a_{\mu\nu}F^a_{\lambda\rho} = 4\int d^3x\,dt \sum_{a,i} E_i^a B_i^a .$$

This action functional is called topological because it does not involve the space-time geometry. Moreover, the topological density function $\sum E_i^a B_i^a$ is a total divergence:

Abelian case. By using $\vec{E} = -\mathrm{grad}\,\phi - \frac{\partial}{\partial t}\vec{A}$ and $\vec{B} = \mathrm{rot}\,\vec{A}$ one has

$$\mathrm{div}\left(-\phi\vec{B} + \vec{A}\times\vec{E}\right) - \frac{\partial}{\partial t}\,\vec{A}\cdot\vec{B} =$$

$$= -\mathrm{grad}\,\phi\cdot\vec{B} + \mathrm{rot}\vec{A}\cdot\vec{E} - \vec{A}\cdot\cancel{\mathrm{rot}\vec{E}} - \frac{\partial}{\partial t}\vec{A}\cdot\vec{B} - \vec{A}\cdot\cancel{\frac{\partial}{\partial t}\vec{B}} = 2\,\vec{E}\cdot\vec{B}.$$

Here is the same calculation using space-time index notation:

Let $\xi^\mu := \epsilon^{\mu\nu\lambda\rho}A_\nu F_{\lambda\rho}$. Then $\partial_\mu \xi^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}$.

Hence, by Stokes' Theorem the topological action vanishes for space-time manifolds without boundary (or electromagnetic fields that vanish at infinity).

Remark. Adding S_{top} to the Maxwell action breaks parity and time-reversal invariance. S_{top} has no effect on the equations of motion, but it does change the Poisson brackets.

Nonabelian case. Consider the vector field $\xi^\mu := \epsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\left(A_\nu \partial_\lambda A_\rho + \frac{2}{3} A_\nu A_\lambda A_\rho\right)$.

Its divergence
$$\begin{aligned}\partial_\mu \xi^\mu &= \epsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\left(\partial_\mu A_\nu \cdot \partial_\lambda A_\rho + 2\,\partial_\mu A_\nu \cdot A_\lambda A_\rho\right)\\ &= \epsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\left(\partial_\mu A_\nu\right)\left(\tfrac{1}{2}\partial_\lambda A_\rho - \tfrac{1}{2}\partial_\rho A_\lambda + [A_\lambda, A_\rho]\right)\\ &= \frac{1}{4}\,\epsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial_\lambda A_\rho - \partial_\rho A_\lambda + 2\,[A_\lambda, A_\rho]\right)\\ &= \frac{1}{4}\,\epsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\, F_{\mu\nu} F_{\lambda\rho} \qquad \text{since } \epsilon^{\mu\nu\lambda\rho} A_\mu A_\nu A_\lambda A_\rho = 0\end{aligned}$$

is equal to the topological density. Hence again, by Stokes' Theorem, we may convert the topological action to a surface integral:

$$S_{top} = \frac{1}{4}\int_M d^3x\,dt\; \epsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\, F_{\mu\nu} F_{\lambda\rho} = \int_M d^3x\,dt\; \partial_\mu \xi^\mu = \int_{\partial M} \mathrm{Tr}\left(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A\right).$$

This surface integral is referred to as the non-Abelian Chern–Simons action.

Now assume that the space-time manifold M is closed or the field strength tensor $F_{\mu\nu}$ vanishes at infinity. In the present (non-Abelian) case it does not follow that S_{top} vanishes. Rather, by taking A at infinity to be pure gauge, i.e. $A_\mu \xrightarrow{|r|\to\infty} g^{-1}\partial_\mu g$ (so that $F_{\mu\nu} \xrightarrow{|r|\to\infty} 0$) one finds

$$S_{top} = \int_{\partial M} \mathrm{Tr}\left((g^{-1}dg) \wedge d(g^{-1}dg) + \tfrac{2}{3}(g^{-1}dg)^{\wedge 3}\right) = -\frac{1}{3}\int_{\partial M} \mathrm{Tr}\,(g^{-1}dg)^{\wedge 3}.$$

This integral computes the "winding number" of the mapping $g : \partial M \to G$.

(If ∂M is a 3-sphere then this is the homotopy invariant $\pi_3(G) = \mathbb{Z}$.)

What is important is that a suitable choice of normalization for Tr makes S_{top} integer-valued. With that choice of normalization we may add θS_{top} for $\theta = \pi$ to the action: $S = S_{YM} + \theta S_{top}$, without breaking parity or time-reversal invariance. Indeed, $\exp(i\pi S_{top}) = \exp(-i\pi S_{top})$ if $S_{top} \in \mathbb{Z}$.